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A problem of stability in phase shift analysis

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Abstract. Even if the phase of a scattering amplitude may be reconstructed uniquely from a certain set of observable quantities, the problem of its stability with respect to small displacements of the data is still open. We present a detailed discussion of the stability of the phase of the πN scattering amplitudes obtained from differential cross section and polarization measurements, using analyticity at fixed momentum transfer and isospin invariance.

1 Introduction

The problem of the determination of the phase of the scattering amplitudes in particle physics from experimental data is well-known and attracted considerable attention over the past decade (see [1] for a review of its status in 1975). In [2, 3], the present author gave a proof that the two independent πN scattering amplitudes $A_+(s,t)$, $B_+(s,t)$ (see [3, 4] for notation) may be determined unambiguously from data on the polarization and differential cross section of the isospin related reactions $\pi^+ p \to \pi^+ p$, $\pi^- p \to \pi^- p$, $\pi^- p \to \pi^0 n$ if one imposes analyticity constraints with respect to the energy at fixed momentum transfer t and at fixed energy with respect to the angle. The class of amplitudes which may be reconstructed this way is delimited by some generally acceptable or experimentally verifiable restrictions, which we recall below. The approach of [2, 3] allows one to settle the issue of stability which has been raised a certain number of years ago [1, 5, 6] in relation with other arguments for uniqueness. From the outset, one should point out that both uniqueness and stability are possible only if one admits that observables are available with finite errors at all energies above threshold, including the unphysical part of the cut, in some range of values $0 \geqq t \geqq t_0$. The constraining power of analyticity at fixed t rests on this hypothesis.

Granted this, the construction of amplitudes of [2, 3] is stable in an intuitively clear manner, and precise statements on this item are provided by [3]. Their proof is not difficult and was omitted in [3]. In this paper, we take up again the question of stability, and give it a more detailed treatment than was possible in [3]. In particular, we present the proofs that are missing there.

In Sect. 2, we recall the main results of [2, 3] and specify the class of amplitudes for which they hold. In Sect. 3, we formulate the problem of stability. Section 4 contains a discussion of stability at "small" $|t|$ (in a sense that will be given in Sect. 3) where it is easier and extend the results to larger $|t|$ in Sect. 6. In Sect. 5, we discuss the way to judge the interval of validity in t of the phase reconstructed by means of analyticity constraints.

2 Restrictions on the amplitudes and results on uniqueness

We call $\mathscr{U}$ the class of pairs of functions $A_+(s,t)$, $B_+(s,t)$ with the following properties: (i) $A_+(s,t)$, $B_+(s,t)$ are holomorphic in two variables in a domain D of $\mathbf{C}^2$, that includes a set $(|t| \leqq |t_0|) \times$ (corresponding cut s plane), for some t_0, with $|t_0| < 4m_\pi^2 \equiv 4\mu^2$; (ii) at each t, $|t| < t_0$, $|A(s,t)| < C|s|^{N(t)}$, with, e.g. $N(t) < 2$ (see [7–8] for a description of the axiomatic domain of analyticity); (iii) $A_+(s,t)$, $B_+(s,t)$ are continuous together with their partial derivatives up to order two as functions of four real variables at points of the cuts; (iv) at each $t \in \mathbf{R}$, $t \geqq t_0$, the phases $\Phi_A(s)$, $\Phi_B(s)$ of A_+, B_+ may be defined by continuity along the cuts, or by means of small excursions in the complex s plane at some finite number of points and are bounded: $|\Phi_{A,B}(s)| < C(t)$.

The class of holomorphic functions of s, obtained by restricting any member of a pair of $\mathscr{U}$ to a fixed value of t is called $P(s_L, s_R)$, where s_L, s_R are the "tips" of the cuts in the s plane. The functions of class P have several properties that are useful in the following and that are summarized in Appendix A of [3]. We also assume then:

(v) At each fixed t, the amplitudes $h_+(v,t)$, $h_-(v,t)$

(see [2–4, 9] for definitions) are of class $P(-\nu_{th}, \nu_{th})$, $\nu_{th} = \mu + t/(4m)$, $\nu = (s-u)/(4m)$. The amplitudes h_+, h_- bear a direct relation to the observables: $|h_+(\nu,t)|^2$, $|h_-(\nu,t)|^2$, $|h_+(\nu,t) - h_-(\nu,t)|^2$ are known for $|\nu| > \bar{\nu}(t)$; $\bar{\nu}(t)$ is the backward direction of the physical region. We call these quantities $f_+^2(\nu,t)$, $f_-^2(\nu,t)$, $f_0^2(\nu,t)$. For $\nu_{th} < |\nu| < \bar{\nu}(t)$, we show in [2, 3] that the following combinations of $h_+(\nu,t)$, $h_-(\nu,t)$ are known (with errors) by analytic continuation of the observables off the physical region:

$$K_+(\nu,t) \equiv h_+(\nu,t) h_-^*(-\nu,t) \tag{2.1}$$

$$K_-(\nu,t) \equiv h_-(\nu,t) h_+^*(-\nu,t) \tag{2.2}$$

$$K_0(\nu,t) \equiv -h_0(\nu,t) h_0^*(-\nu,t) \tag{2.3}$$

and thus:

$$\bar{K}(\nu,t) = K_+(\nu,t) + K_-(\nu,t) - K_0(\nu,t). \tag{2.4}$$

It is sufficient to prescribe these complex functions for $\nu > \nu_{th}$. Their values for $\nu < -\nu_{th}$ are obtained by appropriate complex conjugation (see [2, 3]). We call the set of six functions $(f_+, f_-, f_0, K_+, K_-, K_0)$ the observables $o(t)$. The method of reconstruction of [2] requires also knowledge of the combination:

$$F(\nu,t) = (h_+(\nu,t) h_-^*(\nu,t) - h_-(\nu,t) h_+^*(\nu,t))/4i, \quad |\nu| > \bar{\nu}(t) \tag{2.5}$$

$$= (h_+(\nu,t) h_+^*(-\nu,t) - h_-(\nu,t) h_-^*(-\nu,t))/4i, \quad \nu_{th} < |\nu| < \bar{\nu}(t) \tag{2.6}$$

which is the area of the isospin triangle of the three amplitudes h_+, h_-, h_0 and its analytic extension at fixed s to the unphysical region. The quantity $F^2(\nu,t)$ may be obtained directly from the observables for all ν and it is assumed that analyticity at fixed energy and knowledge of the forward amplitude allows the determination of the sign of $F(\nu,t)$ (see [2]). The optical theorem ensures knowledge of the forward amplitude (see [5, 10, 3]). With this, the class $\mathscr{U}$ of amplitudes is also restricted through the following conditions:

(e1) there is no point (ν,t) with $|\nu| \geqq \bar{\nu}(t)$, $0 \geqq t \geqq t_0$, so that f_+, f_-, $f_0(\nu,t)$ vanish simultaneously; there is no point (ν,t), $\nu_{th} < \nu < \bar{\nu}(t)$, so that K_+, K_-, $K_0(\nu,t)$ vanish simultaneously;

(e2) there exists a constant $c_+ > 0$ and an energy s_+ so that $|h^+(\nu,t)| > c_+|\nu|$ for $|\nu| > \nu(s_+,t)$, $t_0 \leqq t \leqq 0$.

In (e2),

$$h^+(\nu,t) = h_+(\nu,t) + h_-(\nu,t) \tag{2.7}$$

and $|h^+(\nu,t)|^2$ may be easily obtained from f_+^2, f_-^2, f_0^2.

(e3) The forward amplitude of elastic scattering $h_+(\nu, t=0)$ has no symmetrical zeros in the complex ν plane (i.e. if $h_+(\nu,0)=0$, then $h_+(-\nu,0) \neq 0$ and $h_+(0,0) \neq 0$);

(e4) there exist constants c_s, ν_0, so that, for $|\nu| > \nu_0$, $t_0 \leqq t \leqq 0$, $|A^+(\nu,t)|/|C^+(\nu,t)| < c_s$.

In (e4), the amplitudes A^+, C^+ are (t channel) isospin even combinations of A, C (see [4]). Concerning C^+, one also assumes about its phase $\Phi_C(\nu,t)$:

(r1) As $\nu \to \infty$, $\Phi_C(\nu,t) \to \Phi_\infty(t)$, uniformly with respect to t in $(t_0, 0)$. This exhausts the definition of the class $\mathscr{U}$; it is shown in [2, 3] that there exists an interval $(t_1, 0) \subset (t_0, 0)$, such that the amplitudes (h_+, h_-) may be obtained uniquely from the observables. The value $t_1(o)$ is a priori unknown, it depends on the observables $o(t)$, $0 \geqq t \geqq t_0$ and one needs some way to determine it (see below).

The first step of the reconstruction is the determination of the ratio:

$$R(\nu) = \frac{h_-(\nu)}{h_+(\nu)} \tag{2.8}$$

from $o(t)$:

$$R(\nu) = \frac{f_+^2 + f_-^2 - f_0^2 + 4iF(\nu)}{2f_+^2}; \quad |\nu| > \bar{\nu}(t) \tag{2.9}$$

$$R(\nu) = \frac{2K_-}{\bar{K} + 4iF} \quad \nu_{th} < \nu < \bar{\nu}(t). \tag{2.10}$$

It is then argued in [2, 3] that the zeros and poles of $R(\nu,t)$ determine all the zeros of $h_+(\nu,t)$, $h_-(\nu,t)$ if $t_1(o) < t < 0$, where, we stress, $t_1(o)$ depends on the chosen element of $\mathscr{U}$. It is this determination which makes the main object of the discussion below on stability. If the zeros ν_i of $h_+(\nu,t)$ are known, then the algebra given in Appendix A reconstructs $h_+(\nu,t)$, $t_1(o) < t < 0$.

The problem of stability is discussed within a still smaller class of amplitudes, which we call $\mathscr{S}$. The class $\mathscr{S}$ consists of those elements of $\mathscr{U}$ which verify:

$$\text{(s1)} \quad |h_0(\nu,t)| < c_0|\nu|^\gamma \tag{2.11}$$

for $|\nu| > \nu_0$, $t \in (t_0, 0)$, and some γ, $0 < \gamma < 1$, $c_0 > 0$;

(s2) the zeros of $h_+(\nu,t)$, $h_-(\nu,t)$ for $|\nu| > \nu_{th}$, $0 > t > t_0$ are isolated, i.e. if $h_+(\nu_N, t_N) = 0$, $\nu_N, t_N \in \mathbf{R}$, there exists $r > 0$, so that $h_+(\nu,t) \neq 0$ in $0 < |\nu - \nu_N| < r$, $0 < |t - t_N| < r$, $\nu, t \in \mathbf{R}$.

(s3) there is no value of t in $(t_0, 0)$ such that $h_+(\nu_{N1}, t) = 0$, $h_-(\nu_{N2}, t) = 0$, $\nu_{N1}, \nu_{N2} \in \mathbf{R}$, $\nu_{N1,2} > \nu_{th}$;

(s4) the following inequalities hold for large $|\nu| \in \mathbf{R}$, $0 \geqq t \geqq t_0$:

$$\left|\frac{\nu}{f^+}\frac{df^+}{d\nu}\right| < \text{const}; \quad \left|\nu \frac{df_0}{d\nu}\right| < \text{const}|\nu|^\gamma \tag{2.12}$$

$$\left|\frac{dR}{d\nu}\right| < C_1 |\nu|^{\gamma - 2}. \tag{2.13}$$

The last inequality is not a consequence of the former two; it serves also to bound the quantity $dF/d\nu$ which is only accessible with difficulty.

(s5) for any $m > 0$ and sufficiently small, at those points ν, for which $|R(\nu,t)| < m$, it is true that $|R(\nu,t)| > C|\nu - \nu_1|^\beta$, for some $C, \beta > 0$ and $\bar{\nu} \leqq |\nu_1|$. The same is true for $1/R(\nu,t)$. The constants C, β may be chosen uniformly in $\mathscr{S}$.

2 Formulation of the problem of stability

3.1 General discussion

We call $\mathcal{M}(\mathcal{S})$ the set of observables pertaining to amplitudes of class $\mathcal{S}$, for $0>t>t_0$, $\nu_{th}\leqq|\nu|$. It is convenient to regard an element of $\mathcal{M}(\mathcal{S})$ as a set of *eight* functions $(K_+, K_-, K_0, f_+, f_-, f_0, F_1, F_2)$, where we have appended the areas F_1, F_2, (2.5, 2.6) of the isospin triangles for $\nu_{th}<\nu<\bar{\nu}$, $\bar{\nu}\leqq|\nu|$, in turn to the six functions already discussed. Clearly, F_1^2, F_2^2 may be expressed algebraically in terms of K_+, K_-, K_0 or f_+, f_-, f_0. We write $o(t)$ if the observables are restricted to a fixed value of t; otherwise, $o\equiv\{o(t), 0>t>t_0\}$. We call $\mathcal{L}$ the operator of reconstruction, leading from $\mathcal{M}(\mathcal{S})$ back to $\mathcal{S}$, for $0\geqq t\geqq t_1(o)$. Stability means that, if $o_n\to o$ in some sense in $\mathcal{M}(\mathcal{S})$, then $\mathcal{L}(o_n)\to\mathcal{L}(o)$ (in some sense). This is not the way in which the problem appears in reality, since the set $\mathcal{M}(\mathcal{S})$ is "thin" in any topology that mirrors appropriately the effect of experimental errors. We start then from an enlarged space of observables, at fixed t, which we call $\Delta O(t)$, made up of all octuplets of functions $(\Delta K_{+,-,0}, \Delta F_{+,-,0}, \Delta F_{1,2})$ continuous and with a continuous derivative on their appropriate domains of definition, with a norm given below, and try to define an inverse operator $\mathcal{L}_t$ on $\Delta O(t)$, leading from $\Delta o(t)\in\Delta O(t)$ to a space $H(t)$ of pairs of functions $h_+(\nu,t)$, $h_-(\nu,t)$, holomorphic in the cut ν plane, with some topology (given further down). This operator reduces to $\mathcal{L}$ when $\Delta o(t)$ is obtained from $o+\Delta o$ in $\mathcal{M}(\mathcal{S})$. In these definitions, we implied that $\Delta o=0$ in $\Delta O(t)$ corresponds to an element of $\mathcal{M}(\mathcal{S})$. The stability problem is to examine the conditions under which $\mathcal{L}_t:\Delta O(t)\to H(t)$ is continuous at $\Delta o=0$.

In fact, as we shall see, $\mathcal{L}_t$ is not defined over the whole $\Delta O(t)$, but again on a "thin" subset of it $\Delta O_P(t)$, which contains $\Delta o=0$, if $t\in(t_1(o),0)$. One requires then a "fitting" procedure, which we shall leave quite arbitrary, to replace an element of $\Delta O(t)$ by one in $\Delta O_P(t)$.

3.2 Properties of the class $\mathcal{S}$

We give first a statement showing how $t_1(o)$ may be estimated from $o\in\mathcal{M}(\mathcal{S})$. Its conditions are ideal, in that $o(t)$ is not affected by any errors. Then $R(\nu)$, (2.8) may be obtained in the whole complex plane exactly by analytic continuation from its known boundary values, (2.9, 2.10), and its zeros and poles correspond to zeros of $h_+(\nu,t)$, $h_-(\nu,t)$, in so far the latter do not have coincident zeros. The statement is also valid if $o\in\mathcal{M}(\mathcal{U})$ (see [2, 3]); it describes what happens if we ignore the possible coincident zeros of h_+, h_-:

Theorem 3.1. *Let $h_{+a}(\nu,t)$, $h_{-a}(\nu,t)$ be amplitudes obtained from $o(t)$, $o\in\mathcal{M}(\mathcal{S})$, using only the zeros and poles of $R(\nu,t)$, for $t_0<t<0$. If, for all t in some interval $(\tau,0)\subset(t_0,0)$:*

(i) *there exists $\rho>0$, independent of t, so that the zeros of $C_a^+(\nu,t)$ in the ν plane stay contained in a disk $|\nu|<\rho$;*

(ii) *the quantities*

$$m_1(t)=\max\frac{|\nu B^+(\nu,t)|}{|C^+(\nu,t)|} \tag{3.1a}$$

estimated on a circle of radius $\rho_1>\rho$ and

$$m_1'(t)=\sup_{|\nu|>\rho_1,\nu\in\mathbf{R}}\frac{|\nu B^+(\nu,t)|}{|C^+(\nu,t)|} \tag{3.1b}$$

are such that:

$$m_2(t)=\max(m_1(t), m_1'(t)) \tag{3.2}$$

obeys the inequality:

$$\frac{\sqrt{|t|}\,m_2(t)}{2m}<1-\delta \tag{3.3}$$

for some $\delta>0$ ($m=$ mass of the nucleon)

(iii) *the plase $\Phi(\nu,t)$ of $C_a^+(\nu,t)$ is, for ν sufficiently large, contained in an interval $(\Phi_0,\pi-\Phi_0)$, $\Phi_0>0$ then $h_{+a}(\nu,t)$, $h_{-a}(\nu,t)$ coincide with h_+, h_- on $(\tau,0)$.*

The proof is given in [2]. We refer to conditions (i)–(iii) above as conditions (V) and shall use them below in relation with error affected data (see Sect. 5). Conditions (V) ensure that, for $|\nu|$ large enough in the complex ν plane, $|h^+(\nu,t)|>\text{const}|\nu|$; this means that $h^+\neq0$ there and, in particular, that h_+, h_- do not have coincident zeros in that region. Clearly, what is ensured is much stronger than what is required and conditions (V) may well be violated, while $h_+(\nu,t)$, $h_-(\nu,t)$ still fail to have coincident zeros.

We denote by $\overline{t_1(o)}$ the infimum of the values τ, for which conditions (V) hold. Thus, $|\overline{t_1(o)}|<|t_1(o)|$, and sentences involving $t_1(o)$, $\overline{t_1(o)}$ in the following are meaningful in the limit of zero errors. The notation $\overline{t_1(o)}$ is new with respect to [2, 3].

Now, even if $t\in(\overline{t_1(o)},0)$, but $o\in\mathcal{M}(\mathcal{U})$, we cannot prevent the occurrence that h_+, h_- acquire zeros from infinity, as we move down in t, at noncoincident positions. This would prevent a good control of the number of zeros in the complex ν plane of $h_+(\nu,t)$, $h_-(\nu,t)$ and is an obstacle to stability. The condition $o\in\mathcal{M}(\mathcal{S})$ removes precisely this.

Namely, if $R(\nu,t)\neq0$, $1/R(\nu,t)\neq0$ for $|\nu|>\nu_{th}$, we may define the phase of $R(\nu,t)$ unambiguously by continuity from thresholds to the right and to the left and obtain:

$$n_\delta=[(\delta_R(+\infty)-\delta_R(\nu_{th}))+(\delta_R(-\nu_{th})-\delta_R(-\infty))]/\pi. \tag{3.4}$$

The existence of the limits $\delta_R(+\infty)$, $\delta_R(-\infty)$ is ensured by the fact that $|R(\nu,t)|\to1$ as $|\nu|\to\infty$, $\nu\in\mathbf{R}$, for all $t\in(t_0,0)$ (according to (e2), (s1)). With this, we have:

Theorem 3.2. *If $o \in \mathcal{M}(\mathcal{S})$, $t \in (t_0, 0)$*

$$n_\delta = n_+ - n_- \tag{3.5}$$

where n_+, n_- are the numbers of zeros of $h_+(\nu,t)$, $h_-(\nu,t)$ in the interior of the cut ν plane.

Proof. The assumption that h_+, $h_- \in P(-\nu_{th}, \nu_{th})$ implies that (see Appendix A of [3, 10]):

$$R(\nu,t) = \frac{B_-(\nu,t)}{B_+(\nu,t)} E_R(\nu,t) \tag{3.6}$$

where $B_-(\nu,t)$, $B_+(\nu,t)$ are Blaschke products containing the zeros of $h_-(\nu,t)$, $h_+(\nu,t)$ in turn and the nucleon poles, and $E_R(\nu,t)$ is the outer function with modulus $|R(\nu)|$ along the cuts. Now, (2.9) and the assumptions (e2), (s1) imply that, for large $\nu \in \mathbf{R}$:

$$\| R(\nu)| - 1| < |R(\nu) - 1| < \text{const}(f_0/f_+)^2 < C|\nu|^{2\gamma-2}. \tag{3.7}$$

Equation (3.7) and the differentiability of $|R|$ for large $|\nu|$, $\nu \in \mathbf{R}$ imply (see Appendix B) that:

$$E_R(\nu) \to \exp(i\Phi_0), \quad \text{as} \quad |\nu| \to \infty \tag{3.8}$$

uniformly in all directions of the ν plane, $0 < \arg \nu < \bar{\pi}$ (and to the conjugate value for $-\pi < \arg \nu < 0$). The Blaschke products $B_+(\nu,t)$, $B_-(\nu,t)$ contain each a finite number of zeros and thus their ratio tends to finite limits as $|\nu| \to \infty$, above or below the cuts; these limits are the same and uniform in all directions in the upper and lower half plane in turn, for each $t \in (t_0, 0)$. It follows that:

$$R(\nu) \to 1, \quad \text{as} \quad |\nu| \to \infty \tag{3.9}$$

uniformly in all directions, $t \in (t_0, 0)$. The theorem of the variation of the argument establishes then (3.5).

Now, the zeros of $B_+(\nu,t)$, $B_-(\nu,t)$ may wonder to infinity and their number may change as we move down in t in $(t_0, 0)$. This is possible, because the phases of $h_+(\nu,t)$, $h_-(\nu,t)$ as $|\nu| \to \infty$, $\nu \in \mathbf{R}$, are not controlled in the whole interval $(t_0, 0)$. However, (2.11) implies, with the Phragmén–Lindelöf theorem, that $h_0(\nu)/\nu^r|$ $\langle$const for $|\nu| \rangle r \rangle 0$ in the complex plane. Thus, if $t \in (\overline{t_1(o)}, 0)$, in the class $\mathcal{S}$, $|h_{+,-}(\nu)| \rangle$ const$|\nu|$ for $|\nu|$ large enough, $|\nu| \rangle \nu_0(t)$, in the complex plane and consequently, no zeros may migrate to infinity in this range. Therefore, we have:

Theorem 3.3. *If $t \in (\overline{t_1(o)}, 0)$, changes of n_+, n_- may occur only through zeros of $R(\nu,t)$, $1/R(\nu,t)$, for $\nu_{th} \langle |\nu| \langle \nu_0$, for some finite ν_0.*

The continuity of the amplitudes in four real variables at points of the cut ensures that zeros may leave the cut ν plane only through zeros of h_+, h_- at $|\nu| \rangle \nu_{th}$, $\nu \in \mathbf{R}$. Because of (e2), (s1), such zeros may occur only in a finite interval (ν_{th}, ν_0) of the cut.

Now, conditions (s2), (s3) show that, if $o \in \mathcal{M}(\mathcal{S})$, $n_\delta(t)$ is a piecewise constant function having jumps at different values of t, $t \in (\overline{t_1(o)}, 0)$ and that, knowing $n_\delta(t)$, we may count the number of zeros of $h_+(\nu,t)$, $h_-(\nu,t)$, knowing it at $t = 0$.

If one uses the fact that $\sigma_{\text{tot}}(\pi^\pm p) \neq 0$ for all ν, (s2) implies that there exists an interval $(t_2, 0)$ of values of t, where both $h_+(\nu,t) \neq 0$, $h_-(\nu,t) \neq 0$ for $|\nu| > \nu_{th}$, $\nu \in \mathbf{R}$. With Theorem 3.2, the number of zeros of $h_+(\nu,t)$, $h_-(\nu,t)$ is the same on $(t_2, 0) \cap (\overline{t_1(o)}, 0)$ as at $t = 0$ and this simplifies the discussion considerably. The phrase "small" $|t|$ means in the following $t \in (t_2, 0)$. The case of "large" $|t|$ is deferred to Sect. 6.

3.3 The inverse mapping $\mathcal{L}_t$ for small $|t|$

It is of some advantage to replace the cut ν plane by the unit disk of the z plane, through the mapping (A.2). We take then in $\Delta O(t)$ the norm ($z = e^{i\theta}$):

$$\begin{aligned} \|\Delta o\| = \max\Bigg[& \sup_\theta |\Delta K_{+,-,0}(\theta)|, \\ & \cdot\sup_\theta \left|\frac{d}{d\theta} \Delta K_{+,-,0}(\theta)\right|, \quad \sup_\theta |\Delta f_{+,-,0}(\theta)| \\ & \cdot\sup_\theta \left|\frac{d}{d\theta} \Delta f_{+,-,0}(\theta)\right|, \quad \sup_\theta |F_1(\theta)/\theta|, \\ & \cdot\sup_\theta |F_2(\theta)|, \quad \sup_\theta \left|\frac{d}{d\theta} \Delta F_{1,2}(\theta)\right| \Bigg] \end{aligned} \tag{3.10}$$

where the suprema are to be taken on the images of the various domains of definition on $|\nu| \rangle \nu_{th}$ of the functions in question. Since $d\theta/d\nu \sim 1/\nu^2$ for $|\nu| \to \infty$, $\theta \to \pm\pi/2$, (3.7) assumes a very smooth variation of the observables as $\nu \to \infty$. We have divided off a factor θ in the supremum over F_1, because $F_1(\theta) \sim \theta$ as $\theta \to 0$.

We isolate first a subset $\Delta O_A(t)$ of $\Delta O(t)$, on which the correct algebraic relations of F_1, F_2 to the other observables hold; $\Delta O_A(t)$ inherits the distance from $\Delta O(t)$.

If $t \in (t_2, 0)$, $1/R(\nu,t) \neq 0$ for all ν and (2.9, 2.10) map continuously a sufficiently small ball $\|\Delta o(t)\| < \delta$ in $\Delta O_A(t)$ into the space $\Delta\mathcal{R}$ of functions $\Delta R(\theta)$, continuously differentiable on $|z| = 1$, with the norm:

$$\|\Delta R\|_R = \max\left[\sup_\theta |\Delta R|, \sup_\theta \left|\frac{d}{d\theta} \Delta R\right| \right]. \tag{3.11}$$

Verifying the continuity of the mapping $\Delta O_A \to \Delta\mathcal{R}$ makes use of the estimates (2.12, 2.13).

For further work, it is useful to notice that the quantities $f^+ + \Delta f^+$, $f_0 + \Delta f_0$, $R + \Delta R$ satisfy the same inequalities (2.12, 2.13) if $\|\Delta o\| <$ const.

It is not true that any sufficiently small ball in $\Delta\mathcal{R}$ is covered by the values of the mapping (2.9, 2.10) since, e.g. $\Delta R \to 0$ as $\theta \to \pm\pi/2$, because of (s1), (e2). However, we stick to (3.11) for simplicity.

Since $t \in (t_2, 0)$, the function $R(\nu) \equiv R(o; \nu)$ obtained from $o(t)(\Delta o(t) \equiv 0)$ obeys $|R(\nu)| > m > 0$. Now, for all ΔR in $\|\Delta R\| < m/2$, that are images of points in $\Delta O_A(t)$, the variation of the phase of $R(\nu) + \Delta R(\nu) \equiv R(o + \Delta o; \nu)$ as we move along $|z| = 1$ is the same as for $R(o)$. This follows from the fact that $R(o + \Delta o; \nu) \to 1$

as $|v| \to \infty$; thus, in the R plane, the curve $R(o + \Delta o; \theta)$, $\theta \in [0, 2\pi]$ is closed and the number of its turns around the origin is insensitive to small changes of the curve. We conclude:

Lemma 1. *If $t \in (t_2, 0)$ and r is sufficiently small, $n_\delta(t)$ is the same for all $\Delta o(t)$ with $\|\Delta o(t)\| < r$.*

The point of Lemma 1 is that zeros of h_+, h_- may be counted even if small errors are present, if we are close enough to an element of $\mathcal{M}(\mathcal{S})$.

We consider now in $\Delta\mathcal{R}$ the subset $\Delta\mathcal{R}_R$ of functions $\Delta R(v)$ so that $(R + \Delta R)(v)$ is meromorphic in the cut v plane and obeys (3.6) with *at most* $n_+, \underline{n_-}$ zeros in B_+, B_- in turn. It is important that: if $t \in (t_1(o), 0)$, the subset $\Delta\mathcal{R}_R$ contains points of the image in $\Delta\mathcal{R}$ of a ball $\|\Delta o\| < r$ in $\Delta O_A(t)$. Indeed, $\Delta o = 0$ belongs to $\Delta\mathcal{R}_R$ and we may obtain other points of $\Delta\mathcal{R}_R$ by, e.g. slightly displacing the zeros of $h_+(v, t)$, $h_-(v, t)$ in the v plane. Let $\Delta O_R(t)$ be the set of those elements in $\Delta O_A(t)$ that lead, via (2.9, 2.10) to points in $\Delta\mathcal{R}_R$. Notice, ΔO_R contains "isolated" subsets where the zeros of h_+, h_- coincide; there, B_+, B_- have less than n_+, n_- zeros. With this, (A.1–A.8) of Appendix A define a natural action of $\mathcal{L}_t$ on $\Delta O_R(t)$.

In general, the amplitudes h_+, h_- obtained this way are neither holomorphic in the cut v plane, because the sum rule (A.9) is violated, nor continuous at $v = \pm\bar{v}(t)$, since (A.10, A.13) are not obeyed. However, they are satisfied at $\Delta o = 0$ and there are changes of the observables which keep them intact, away from $\Delta o = 0$. We call $\Delta O_P(t)$ the subset of $\Delta O_R(t)$, where (A.10, A.13) are fulfilled. We take $\Delta O_P(t)$ as the natural domain of definition of the inverse operator $\mathcal{L}_t$.

In the following, we write occasionally $g(o + \Delta o; v)$ for a function $g(v)$ needed in the algebra of $\mathcal{L}_t$ (e.g. E^1_+, B_+, etc.) and obtained from error affected observables $o + \Delta o$; alternatively, we shall write $g(o + \Delta o; v) \equiv \tilde{g}(v)$, but $g(o; v) \equiv g(v)$.

We define now $P_1(-v_{th}, v_{th})$ as the set of functions holomorphic in the whole cut v plane and obeying the same constraints as the class $P(-v_{th}, v_{th})$, except for the continuity of the derivative on the cuts, which we replace by Hölder continuity of the function of any order less than unity. Then, we have the following:

Lemma 2. *The amplitudes $h_+(v, t)$, $h_-(v, t)$ obtained through the action of $\mathcal{L}_t$ on $\Delta o \in \Delta O_P(t)$ belong to $P_1(-v_{th}, v_{th})$.*

Proof. The dispersion integrals in (A.1–A.8) are all performed over functions with continuous derivative, and thus the result is Hölder continuous of any order less than unity (see, e.g. [12]). The continuity at $v = \pm\bar{v}(t)$ is ensured through (A.10–A.13). We only have to verify that the phase is bounded, as $|v| \to \infty$. But $vL_A(v)$ and the phase of $B_+(v)$ are manifestly bounded and thus, according to (A.8), we only have to verify that the phase of $E^1_+(v; o + \Delta o)$ is bounded. To this end, we use the fact that $o \in \mathcal{M}(\mathcal{S})$, so that the phase of $\ln E^1_+(v; o)$ is bounded (cf. (iv, v) in the definition of $\mathcal{U}$), and compare

$$\ln(E^1_+(v; o + \Delta o)/E^1_+(v; o)) = \frac{(v_{th}^2 - v^2)^{1/2}}{\pi}\left[\int_{-\infty}^{-v} + \int_{v}^{\infty}\right]\frac{\ln(1 + \Delta f/f_+(v'; o))dv'}{(v' - v)(v'^2 - v_{th}^2)^{1/2}}. \tag{3.12}$$

In the second integral, perform the change of variables $v'_1 = 1/v'$, $v_1 = 1/v$ so that one is led to analyze the behaviour of the expression:

$$v_1 \int_0^{1/v} \frac{\ln(1 + \Delta f/f_+(v'_1; o))}{(v_1 - v'_1)(1 - v_{th}^2 v_1'^2)^{1/2}} \tag{3.13}$$

for v_1 near zero. The integrand vanishes at $v'_1 = 0$, and its derivative is bounded there, because of the finiteness of $\|\Delta o\|$ and of condition (2.12) on df^+/dv. Therefore, the integral itself is bounded and thus the expression (3.12) is a constant as $v \to \infty$, which establishes our claim.

To complete the specification of the problem, we give a norm in the image space H of pairs of functions $h_+(o + \Delta o; v)$, $h_-(o + \Delta o; v)$ obtained through the action of $\mathcal{L}_t$ on $\Delta O_P(t)$. Namely, we define first a scale function $E_S(v)$ through the requirements: (i) $E_S(v)$ is holomorphic and without zeros in the cut v plane; (ii) $|h_+(v)|/|E_S(v)| \to 1$ as $|v| \to \infty$, $v \in \mathbf{R}$; (iii) $|E_S(v)| > \text{const} > 0$, as $|v| \to v_{th}$, $v \in \mathbf{R}$; (iv) $|E_S(v)|$ increases for $|v| > v_{th}$. Further, let D_γ denote the interior of two ellipses with foci at $(-\bar{v}(t), -v_{th})$, $(v_{th}, \bar{v}(t))$ and of large semiaxis $|\bar{v} - v_{th}|/2 + \gamma$ and F_γ the complement of $\{D_\gamma \cup (|v \pm v_B| < r)\}$ for some small r. Then, we define:

$$\|(h_+, h_-)\|_H = \max\left[\sup_{F_\gamma}\frac{|h_+(v)|}{|E_S(v)|},\quad \sup_{F_\gamma}\frac{|h_-(v)|}{|E_S(v)|}\right]. \tag{3.14}$$

In the next section, we give a theorem establishing the conditions under which $\mathcal{L}_t$ is continuous from $\Delta O_P(t)$ into H.

4 Proof of stability for small $|t|$

This section is concerned mainly with the proof of the following statement:

Theorem 4.1. Let $h_+(v)$, $h_-(v)$ *be a pair of functions in* $P_1(-v_{th}, v_{th})$ *and such that:* (i) *they do not vanish on the cuts* $|v| > v_{th}$, $v \in \mathbf{R}$; (ii) *they have* n_+, n_- *zeros in the complex v plane, in turn;* (iii) *they lead to observables* $o + \Delta o_0$, $\Delta o_0 \in \Delta O_P$; (iv) *the distance between the zeros of h_+ and those of h_- in the v plane is larger than $d > 0$. Then, for any $\varepsilon > 0$ and $\gamma > 0$, both sufficiently small, there exists $\delta(\varepsilon; \gamma)$ so that, for any other pair of functions of class* $P_1(-v_{th}, v_{th})$, $\tilde{h}_+(v)$, $\tilde{h}_-(v)$, *with the same properties* (i)–(iii) *and generating observables* $o + \Delta o$, $\Delta o \in \Delta O_P$, *the condition:* $\|\Delta o - \Delta o_0\| < \delta$ *implies* $\|(\tilde{h}_+ - h_+, \tilde{h}_- - h_-)\|_H < \varepsilon$.

Before proving this, we make some comments and show how this statement settles the issue of stability.

(i) The contents of Theorem 4.1 is intuitively clear: since we have restricted the number of zeros of h_+, h_- and thus have only a finite number of parameters at our disposal (roughly speaking), we do expect $\tilde{h}_+$, $\tilde{h}_-$ to be close to h_+, h_-, a sketch of a proof is also given in [2]. Here we shall be more explicit concerning the dependence $\delta(\varepsilon)$, in order to show that an error analysis is feasible in principle and because this topic is not treated in the literature.

(ii) If $\Delta o_0 = 0$, i.e. $h_+(v,t)$, $h_-(v,t)$ are of class $\mathscr{S}$, we may apply this theorem if $t \in (\overline{t_1(o)}, 0)$ (since only if $t \in (\overline{t_1(o)}, 0)$ do we know the number of zeros of h_+, h_-) and conclude stability if condition (iv) above on the zeros of $h_+(v)$, $h_-(v)$ is fulfilled. Since this is the case at $t = 0$, there exists an interval $(t_S, 0)$ where they still stay apart. Thus:

Theorem 4.2. *There exists an interval* $(t_S, 0) \subset (\overline{t_1(o)}, 0) \cap (t_2, 0)$ *so that* $\mathscr{L}_t: \Delta O_P(t) \to H$ *is stable at* $\Delta o = 0$ *for all* $t \in (t_S(o), 0)$.

(iii) In an (idealized) experiment, we are given the value of $\|\Delta o(t) - \Delta o_0(t)\| \equiv e_0(t)$. If $e_0(t)$ is sufficiently small, we may find the value of ε, the measure of the departure of the possible $\tilde{h}_+$, $\tilde{h}_-$ from h_+, h_- by solving the equation: $\delta(\varepsilon;\gamma) = e_0(t)$, where $\delta(\varepsilon;\gamma)$ is the monotonically increasing function of ε, whose existence is established by Theorem 4.1.

Proof of Theorem 4.1: Writing $B(o + \Delta o; z) \equiv \tilde{B}(z)$, the first task is to bound, for $z = \exp(i\theta)$:

$$|B(\theta) - \tilde{B}(\theta)| = \left|\frac{B_-(\theta)}{B_+(\theta)} - \frac{\tilde{B}_-(\theta)}{\tilde{B}_+(\theta)}\right| \leqq |R(\theta) - \tilde{R}(\theta)| \cdot \frac{1}{|E_R(\theta)|} + |E_R(\theta) - \tilde{E}_R(\theta)| \frac{1}{|E_R(\theta)|} \tag{4.1}$$

with $E_R(\theta)$ of (3.6). The first term is bounded by $\|\Delta R\|_R/m$, $m = \inf_\theta R(\theta) > 0$ and we can make $\|\Delta R\|_R < \varepsilon_1$, if $\|\Delta o\|$ is sufficiently small. Further,

$$\begin{aligned}\left|\frac{\tilde{E}_R(\theta)}{E_R(\theta)} - 1\right| &= \left|\left|1 + \frac{\Delta R(\theta)}{R(\theta)}\right| \cdot \exp\left[\frac{1}{2\pi i}\oint \frac{\cos((\phi - \theta)/2)}{\sin((\phi-\theta)/2)} \cdot \ln\frac{|1 + \Delta R(\phi)/R(\phi)|}{|1 + \Delta R(\theta)/R(\theta)|} d\phi\right] - 1\right| \\ &\equiv \left|\left|1 + \frac{\Delta R(\theta)}{R(\theta)}\right| \exp[-i\Delta\psi(\theta)] - 1\right| \\ &\leqq \frac{|\Delta R(\theta)|}{|R(\theta)|} + |\Delta\psi(\theta)|\end{aligned} \tag{4.2}$$

where $2\pi\Delta\psi(\theta)$ is the real integral in the exponent, and $\Delta R = \tilde{R} - R$. A bound on $\Delta\psi(\theta)$ is obtained in terms of $|\Delta R|$, $|d\Delta R/d\theta|$ by means of the inequality, valid for $|x|$, $|y| < 1$:

$$|\ln(|1 + x|/|1 + y|) \leqq |x - y|/(1 - \max(|x|, |y|)) \tag{4.3}$$

which leads to:

$$\begin{aligned}|\Delta\psi| &\leqq (C_1 \sup|d\Delta R/d\theta| + 2C_2/m \sup|\Delta R(\theta)|)/m \\ &\leqq \text{const}\,\|\Delta R\|_R \leqq \text{const}\ \varepsilon_1.\end{aligned} \tag{4.4}$$

In the derivation of (4.4), we needed to show that a constant C_2 exists so that, for all θ:

$$\oint |R(\theta) - R(\phi)| \frac{d\phi}{\sin((\theta - \phi)/2)} < C_2. \tag{4.5}$$

This is done by showing that $R(\theta)$ is a Hölder continuous function of θ over the whole circle $|z| = 1$ (i.e. including $\pm\pi/2$), with Hölder index $1 - \gamma$. This follows from (2.13) and the fact that $d\theta/dv \sim 1/v^2$ for large, real v. Thus, the derivative $dR(\theta)/d\theta$ is continuous at all θ, except for $\theta = \pm\pi/2$, where it may diverge like $1/(\pi/2 - \theta)^\gamma$, which establishes our claim. Clearly, in (4.5), C_2 increases indefinitely if $R(\theta) \to \infty$ for some θ_0. Thus, $|B(\theta) - \tilde{B}(\theta)|$ is uniformly less than const $\times\ \varepsilon_1$, where the constant increases indefinitely if $\inf_\theta[R(\theta), 1/R(\theta)] \to 0$.

Next, we show that, if $|B - \tilde{B}| \to 0$, then $|B_+ - \tilde{B}_+| \to 0$ and $|B_- - \tilde{B}_-| \to 0$. Clearly, this can be true only at those values of t where B_+, B_- have no coincident zeros. This is assumption (iv) of Theorem 4.1.

Using $|B_{+,-}(e^{i\theta})| = 1$ and writing $B(z) = B_N(z)B_1(z)$, where $B_N(z)$ contains the nucleon poles, we rewrite our conclusions on (4.1) as: ($z = \exp(i\theta)$):

$$|B_1(z) - \tilde{B}_1(z)| \equiv |\tilde{B}_{1-}(z)B_{1+}(z) - \tilde{B}_{1+}(z)B_{1-}(z)| < \varepsilon_2 \tag{4.6}$$

and the last inequality is valid at all $|z| < 1$ by the maximum modulus principle. We drop in the following the subscript unity to simplify the notation and shall state explicitly when the nucleon pole (which is formally inessential) is reinstated.

We wish to derive from (4.6) a bound on the absolute distance between the zeros $z_{i+,-}$ of $B_{+,-}$ and the corresponding zeros $\tilde{z}_{i+,-}$, of $\tilde{B}_{+,-}$. Clearly, a precise answer depends on the detailed positions of the zeros of B_-, B_+. However, a general (and correspondingly weak) solution may be obtained from the Lemma of Boutroux and Cartan (see [13, p. 46]):

Lemma of Boutroux and Cartan. *Let* $P(z) = \prod_{i=1}^{n}(z - z_i)$. *For any* $H > 0$, *the inequality:*

$$|P(z)| > (H/e)^n \tag{4.7}$$

holds outside at most n circles, the sum of whose radii is at most 2H.

This is applied as follows: first, we notice (4.6) implies, for $i = 1, 2, \ldots, n_-$

$$|\tilde{B}_-(z_{i-})| < \frac{\varepsilon_2}{|B_+(z_{i-})|} < \frac{3^{n_+}\varepsilon_2}{|B_+(z_{i-})|} \tag{4.8}$$

with z_{i-} the n_- zeros of B_- and the last inequality is written for later convenience. Using assumption (iv):

$$|B_+(z_{i-})| = \prod_{j=1}^{n_+} \frac{|z_{i-} - z_{j+}|}{|1 - z_{i-}z_{j+}|} > \prod_{j=1}^{n_+} \frac{|z_{i-} - z_{j+}|}{2^{n_+}} > (d/2)^{n_+} \tag{4.9}$$

so that the bound in (4.8) is independent of i. Now, for any z in $|z| < 1$:

$$|\tilde{B}_-(z)| \equiv \prod_{i=1}^{n_-} \left|\frac{(z - \tilde{z}_{i-})}{(1 - z\tilde{z}^*_{i-})}\right| > \frac{1}{2^{n_-}} \prod_{i=1}^{n_-} |z - \tilde{z}_{i-}|. \tag{4.10}$$

We use now the Boutroux–Carton Lemma to bound from above the "width" of the domain Ω_- where the inequality

$$\prod_{i=1}^{n_-} |z - \tilde{z}_{i-}| < 2^{n_-} \frac{6^{n_+}\varepsilon_2}{d^{n_+}} \equiv \varepsilon_{3-} \tag{4.11}$$

may hold. According to (4.8, 4.9) the zeros z_{i-} of B_- belong to Ω_-. From the Lemma, the sum of the radii of the (at most) n circles where (4.11) may hold does not exceed $2e(\varepsilon_{3-})^{1/n_-}$. A similar domain Ω_+, with a constant ε_{3+}, is obtained containing the zeros of B_+, $\tilde{B}_+$. We now choose ε_1 (and thus ε_2) so small that Ω_+, Ω_- (a) are strictly contained in $|z| < 1$; (b) are disjoint from each other. If, e.g. r_- is the minimal distance of the zeros of B_- to $|z| = 1$, we choose ε_2 so that $2e\varepsilon_{3-}^{1/n_-}(\varepsilon_2) < r_-/2$ and satisfy (a). Further, we let ε_1 be such that $2e(\varepsilon_{3-}^{1/n_-} + \varepsilon_{3+}^{1/n_+}) < d/3$ and satisfy (b). With these choices, we made sure that $B_+(z) \neq 0$, if $z \in \Omega_-$, $B_-(z) \neq 0$ if $z \in \Omega_+$.

Moreover, on the boundary $\partial\Omega_-$, in view of (a):

$$|\tilde{B}_-(z)| \equiv \prod_{i=1}^{n_-} \frac{|z - \tilde{z}_{i-}|}{|1 - z\tilde{z}^*_{i-}|} > \frac{\varepsilon_{3-}}{2^{n_-}} \equiv \frac{6^{n_+}\varepsilon_2}{d^{n_+}}. \tag{4.12}$$

Also, if $z \in \partial\Omega_-$, in view of (b), (a)

$$|B_+(z)| \equiv \prod_{i=1}^{n_+} \frac{|z - z_{i+}|}{|1 - zz^*_{i+}|} > \frac{d^{n_+}}{6^{n_+}} \tag{4.13}$$

and we conclude that, on $\partial\Omega_-$,

$$|\tilde{B}_-(z)B_+(z)| > \varepsilon_2. \tag{4.14}$$

Combining (4.14) with (4.6), applying Rouché's theorem and taking into account (b), we see that the number of zeros of $B_-(z)$ in each connected component of Ω_- is the same as the number of zeros of $\tilde{B}_-(z)$ in that component. Since the diameter of each component is at most $4e\varepsilon_{3-}^{1/n_-}$, we have obtained the desired bound on the absolute distance between the zeros of $B_-(z)$ and those of $\tilde{B}_-(z)$. The argument for B_+ and $\tilde{B}_+$ runs in an obvious manner. We call ε_{4-}, ε_{4+} these bounds, depending on ε_1. As a consequence, we obtain inequalities valid for all ν in the complex plane:

$$|B_-(\nu) - \tilde{B}_-(\nu)| < n_- \max_i \left|\frac{z - z_{i-}}{1 - zz^*_{i-}} - \frac{z - \tilde{z}_{i-}}{1 - z\tilde{z}^*_{i-}}\right| < 2n_-\varepsilon_{4-} \frac{|1 - z(\nu)^2|}{(1 - r_-)^2} \equiv \varepsilon_{5-}(\nu). \tag{4.15}$$

Reinstating the nucleon pole through a Blaschke factor $B_{N-}(\nu)$ invalidates (4.15), but outside a neighbourhood of radius r around $z(-\nu_B)$ it is true that

$$|B_-(\nu) - \tilde{B}_-(\nu)| < 2\varepsilon_{5-}(\nu)/r. \tag{4.16}$$

Clearly, $|B_+(\nu) - \tilde{B}_+(\nu)|$ obeys a similar inequality.

We now proceed to estimate $|h_+(\nu) - \tilde{h}_+(\nu)|$ for all ν in the domain F_γ. The inequalities that follow do not make use of the continuity of h_+ at $\bar{\nu}$ and are thus too weak, although sufficient to establish stability with respect to (3.14). With (A.8), we must bound $T_1(\nu) \equiv |B_+(\nu) - \tilde{B}_+(\nu)|$, which is already available, $T_2(\nu) \equiv |\tilde{E}^1_+(\nu) - E^1_+(\nu)|/|E_S(\nu)|$ and $T_3(\nu) \equiv |\exp[(\nu_{th}^2 - \nu^2)^{1/2}\tilde{L}(\nu)] - \exp[(\nu_{th}^2 - \nu^2)^{1/2}L(\nu)]|$ for all ν in F_γ.

For $T_2(\nu)$, we write:

$$T_2(\nu) = \frac{|E^1_+|}{|E_S|}\left|\exp\left\{(\nu_{th}^2 - \nu^2)^{1/2}\left[\int_{-\infty}^{-\bar{\nu}} + \int_{\bar{\nu}}^{\infty}\right] \cdot \frac{\ln(1 + \Delta f_+(\nu')/f_+(\nu'))}{(\nu_{th}^2 - \nu'^2)^{1/2}(\nu' - \nu)}\, d\nu'\right\} - 1\right| \tag{4.17}$$

where $|\ln(1 + \Delta f_+/f_+)| \leqq \text{const}\,\|\Delta o - \Delta o_0\|/m$. It is expedient to extend $\ln(1 + \Delta f_+/f_+)$ down to ν_{th} by continuity through a function $k(\nu)$, with $k(\nu), dk(\nu)/d\nu = O(\|\Delta o - \Delta o_0\|)$ and otherwise arbitrary, and then subtract this contribution. We call E_C the outer function in the cut ν plane, with modulus $\exp(k(\nu))$, $\nu_{th} < |\nu| < \bar{\nu}$ and unity for $|\nu| > \bar{\nu}$, $\nu \in \mathbf{R}$. For the function $(\tilde{E}^1_+/E^1_+)E_C(\nu)$, the estimates (4.2) may be repeated, using the boundedness of the derivatives $d\Delta f_+/d\theta$ and inequality (2.12), so that, with the maximum modulus principle, we conclude that:

$$\left|\frac{\tilde{E}^1_+(\nu)}{E^1_+(\nu)} E_C(\nu) - 1\right| < \text{const}\,\|\Delta o - \Delta o_0\| \tag{4.18}$$

for all ν in the complex plane, including the cuts. Further, in F_γ:

$$|\ln E_C(\nu)| < C_1 \frac{(\nu_{th}^2 - \nu^2)^{1/2}}{\min_{\partial D_\gamma}|\nu' - \nu|} \cdot \|\Delta o - \Delta o_0\| < C_2(\gamma)\|\Delta o - \Delta o_0\| \tag{4.19}$$

so that:

$$|E_C^{-1} - 1| < \text{const}(\gamma)\|\Delta o - \Delta o_0\|. \tag{4.20}$$

With a similar argument, $|E^1_+|/|E_S|$ is bounded in F_γ by const(γ). This disposes of the term $T_2(\nu)$.

For $T_3(\nu)$, we notice first that, in view of (2.10), $\bar{K} + 4iF_1 \neq 0$ on $(\nu_{th}, \bar{\nu})$ and thus $\text{Im}(K_L - \tilde{K}_L) <$

const $\|\Delta o - \Delta o_0\| / (v^2 - v_{th}^2)^{1/2}$. From (A.5), it follows that, in F_γ:

$$|(v_{th}^2 - v^2)^{1/2} \| L_S(v) - \tilde{L}_S(v)| < \text{const}(\gamma) \| \Delta o - \Delta o_0 \|. \tag{4.21}$$

The quantity $\text{Re}(\tilde{K}_L - K_L)$ involves phase differences of the factors in (A.4) and these vanish as $v \to v_{th}$ like $(v - v_{th})^{1/2}$ (cf. (3.10, A.3)). Notice, the phase of $E_+^1(v)$ has a logarithmic singularity at $v = \bar{v}$, but this does not disturb the bound in F_γ; one verifies that:

$$|(v^2 - v_{th}^2)^{1/2} \| L_A(v) - \tilde{L}_A(v)| < \text{const}(\gamma) \| \Delta o - \Delta o_0 \| + \text{const}(\gamma) g(\varepsilon_{3+}) \tag{4.22}$$

where $g(\varepsilon_{3+})$ comes from the Blaschke factors in (A.5) and vanishes as $\varepsilon_{3+} \to 0$.

Now, putting these bounds together, we may estimate:

$$\frac{1}{|E_S|} |h_+(v) - \tilde{h}_+(v)| < T_1(v) \left| \frac{E_+^1}{E_S} \right| \cdot |\exp[(v^2 - v_{th}^2)^{1/2} L]| + T_2(v) \tilde{B}_+(v) \cdot |\exp[(v_{th}^2 - v^2)^{1/2} L]| + T_3(v) \left| \frac{\tilde{E}_+^1}{E_S} \right| |B_+(v)| \tag{4.23}$$

where the quantities multiplying the factors $T_i(v)$ are bounded by const(γ) in F_γ. Clearly, given ε, γ we may find now ε_{3+} and thus ε_1 and $\| \Delta o - \Delta o_0 \| = \delta$, so that $\| (h_+ - \tilde{h}_+, h_- - \tilde{h}_-) \|_H < \varepsilon$. This ends the proof of Theorem 4.1.

5 Lower bounds on the magnitude of the interval $(t_1(o), 0)$, for small $|t|$

The theorem of stability of the previous section is not yet satisfactory, since it uses knowledge of the quantity $t_1(o)$ for the true observables $o \in \mathcal{M}(\mathcal{S})$. This is, of course, an unknown quantity, if we are simply given $(o + \Delta o)(t)$, for $t \in (t_0, 0)$, $\Delta o(t) \in \Delta O(t)$.

Assuming the latter, the procedure of reconstruction for $t \in (t_2, 0)$ would be as follows: from the known amplitude at $t = 0$, we obtain the value $n = n_+ = n_-$, perform a small displacement of the observables within the errors $e_0(t)$, so that they lie in $\Delta O_R(t)$ and then in $\Delta O_P(t)$ and finally use the formulae of Appendix A. In this process, it may happen that, as $|t|$ increases, we find no elements of $\Delta O_R(t)$ or $\Delta O_P(t)$ in the ball $\| \Delta o \| < e_0(t)$; this means that supplementary zeros have migrated from infinity in both $h_+(v, t)$, $h_-(v, t)$ and it is impossible to reproduce $R(v, t)$ with the minimal number of zeros $n_+ = n_- = n$. It may also be, however, that such zeros appear in both $h_+(v, t)$, $h_-(v, t)$ and nevertheless, the agreement of the "minimal" amplitudes (obtained with $n_+ = n_- = n$ zeros) with the observables is not bad. If the errors are zero, then we verify whether the minimal amplitudes satisfy conditions (V) of Theorem 3.1, and obtain a bound on their range of validity.

Thus, it appears we should extend Theorem 3.1 to the situation when errors are present. This we do in this section, for the case of small $|t|, t \varepsilon (t_2, 0)$, where $R(v) \neq 0$, $1/R(v) \neq 0$, $|v| > v_{th}$, $v \varepsilon \mathbf{R}$, since the discussion is simpler.

The first step is to show that conditions (V) enjoy stability with respect to the distance (3.14) in H. To this end, we denote by $\mathcal{L}_t(r)$ the image in H of a small ball $\| \Delta o \| < r$ in $\Delta O(t)$, as described above. We have then:

Theorem 5.1. *Assume $h_{+a}, h_{-a} \varepsilon \mathcal{L}_t(r)$ satisfies conditions* (V) *of Theorem 3.1 for all t in an interval $(\tau, 0)$. Then, all other elements $(h_+, h_-) \varepsilon \mathcal{L}_t(r)$, so that $\| (h_+ - h_{+a}, h_- - h_{-a}) \|_H$ is sufficiently small, satisfy the same conditions* (V) *for all $t \varepsilon (\tau, 0)$.*

Proof. It is convenient to introduce first a symmetric scale function $E_{SM}(v, t)$, free of zeros in the cut v plane, and with modulus:

$$|E_{SM}(v)| = \frac{|E_S(v)| + |E_S(-v)|}{2}. \tag{5.1}$$

It is true that, for $v \varepsilon F_\gamma$:

$$\frac{|C^+ - C_a^+|}{|E_{SM}(v)|} < \varepsilon. \tag{5.2}$$

Also, because of (3.2a), in condition (ii), for $|v| > v_0$, $v \in \mathbf{R}$:

$$|C_a^+(v, t)| > c_1 |E_{SM}(v)| \tag{5.3}$$

where $c_1 > 0$. Now, it may be shown (see [2, 3]) that conditions (ii), (iii) imply that $|C_a^+(v, t)| > \text{const}\ |v|$, for $|v|$ large enough in the complex plane, and thus that $1/|C_a^+(v, t)|$ is bounded by const$/|v|$. It follows that the inequality:

$$\frac{|E_{SM}(v)|}{|C_a^+(v)|} < \text{const} \tag{5.4}$$

is valid in the whole complex plane, outside the circle $|v| = \rho$ (the constant may be different from $1/c_1$). Therefore, in that domain:

$$\frac{|C^+(v)|}{|E_{SM}(v)|} > \frac{|C_a^+(v)|}{|E_{SM}(v)|} - \varepsilon > \frac{1}{\text{const}} - \varepsilon > 0 \tag{5.5}$$

so that C^+ is also nonvanishing outside $|v| = \rho$, for ε small enough, i.e. condition (i) is fulfilled.

Further, for $v \in F_\gamma$

$$\sqrt{|t|} \frac{|vB^+ - vB_a^+|}{|E_{SM}|} < \text{const} \times \varepsilon \tag{5.6}$$

and, using (5.3), we may evaluate directly

$$\left| \frac{|vB_a^+|}{|C_a^+|} - \frac{|vB^+|}{|C^+|} \right| < \text{const} \times \varepsilon \tag{5.7}$$

so that, if ε is small enough, condition (ii) is also fulfilled. Finally, (5.2) implies a similar inequality for the moduli $|C_a^+|/|E_{SM}|$, $|C^+|/|E_{SM}|$. Then, the cosine rule, together with (5.3), gives:

$$\cos\psi > 1 - \text{const} \times \varepsilon \tag{5.8}$$

where ψ is the angle of C_a^+ with C^+. Thus, condition (iii) is also fulfilled.

We can now state a theorem concerning the approximation of the true amplitudes, which is the main result of this note. It is stated for the interval $(t_2, 0)$, although it is valid down to large $|t|$, as will be shown in the next section.

Theorem 5.2. *Let* $(h_{+a}(v,t), h_{-a}(v,t)) = \mathscr{L}_t(o + \Delta o(t))$, $o \in \mathscr{M}(\mathscr{S})$, *for all* $t \in (\tau, 0) \subset (t_2, 0)$, *and let* $\sup_{t\in(\tau,0)} \|\Delta o(t)\| = \delta_0$. *Assume* (i) *the zeros of* $h_{+a}(v,t)$, $h_{-a}(v,t)$ *in the complex* v *plane stay a distance* $d > 0$ *apart for all* $t \in (\tau, 0)$; (ii) $h_{+a}(v,t)$, $h_{-a}(v,t)$ *satisfy conditions* (V) *of Theorem 3.1 in* $(\tau, 0)$. *Then, if* δ_0 *is sufficiently small,* $h_{+a}(v,t)$, $h_{-a}(v,t)$ *depart from the true amplitudes of class* $\mathscr{S}$, $(h_+(v,t), h_-(v,t)) \equiv \mathscr{L}_t(o)$, *by less than* ε *in the norm (3.14), where* ε *is the root of* $\delta(\varepsilon{:}\gamma) = \delta_0$ *and* $\delta(\varepsilon;\gamma;h_a)$ *is given in Theorem 4.1.* $(t \in (\tau, 0))$.

Proof. We simply put together Theorems 5.1, 4.1 and 3.1. Indeed, we choose first ε_0 so that conditions (V) are satisfied for all h_+, h_- with $\|(h_+ - h_{+a}, h_- - h_{-a})\|_H < \varepsilon_0$. Then, Theorem 4.1 gives us $\delta(\varepsilon_0;\gamma;h_a)$, (if ε_0 is small enough), so that, if $\|\Delta o(t) - \Delta o_0(t)\| < \delta(\varepsilon)$, then $(h_+, \tilde{h}_-) = \mathscr{L}_t(o + \Delta o)$ obeys conditions (V). Assume now $\delta_0 < \delta(\varepsilon_0)$. Then, the action of $\mathscr{L}_t$ on $o \in \mathscr{M}(\mathscr{S})$ leads to functions $h_+(v,t)$, $h_-(v,t)$ satisfying conditions (V) for all $t \in (\tau, 0)$. Comment (iii) in Sect. 4 ends the proof.

6 Stability at larger $|t|$

In this section, we show how to get rid of the assumptions $R(v,t) \neq 0$, $1/R(v,t) \neq 0$ for $|v| > v_{th}$ and obtain essentially the same statements as in Sects. 3, 4. We consider the situation where, at some point (v_0, t_0), $|v_0| > v_{th}$, $h_-(v_0, t_0) = 0$, $h_- \in \mathscr{S}$, and thus $R(v_0, t_0) = 0$. Small errors $\|\Delta o\|$ of the observables alter the position of the zero of $R + \Delta R$ or remove it altogether. According to (s3), if $\eta > 0$ and sufficiently small, $h_-(v, t_0 \pm \eta) \neq 0$, $h_+(v, t_0 \pm \eta) \neq 0$ for all $|v| > v_{th}$, $v \in \mathbf{R}$; it follows that, if r is small enough, the quantity

$$\Delta n \equiv n_\delta(t_0 - \eta) - n_\delta(t_0 + \eta) \tag{6.1}$$

is the same for all $\|\Delta o(t_0 \pm \eta)\| < r$. This defines a minimal change Δn_- of the number of zeros of $h_-(v,t)$ in the complex v plane, as we move down in t past t_0; this is the actual change if $t \in (t_1(o), 0)$. The inverse operator $\mathscr{L}_t$ is defined as in the previous section; we pick out first a subset $\Delta O_{R1}(t)$ of $\Delta O(t)$, on which $R(v,t)$, (2.9, 2,10), has n_+ poles and n_- or $n_- + 1$ zeros; we append now to the definition of $\Delta O_{R1}(t)$ the condition (z): for all $\Delta o(t) \in \Delta O_{R1}(t)$, $(R + \Delta R)(v,t)$ obeys (s5) with the same constants. This restriction allows a control on the behaviour of $\ln(R + \Delta R)(v,t)$ near the zeros of $(R + \Delta R)(v,t)$. Further, we define $\mathscr{L}_t$ on $\Delta O_{P1}(t) \subset \Delta O_{R1}(t)$, as described in Section III.

In the following, we consider only the case $|v_0| > \bar{v}(t)$, so that $f_-(v_0, t_0) = 0$ and shall leave the situation $v_{th} < |v_0| < \bar{v}$ out, since it does not lead to different conclusions but to a more complicated algebra. Also, we discuss only the case $\Delta n_- = 1$. The main concern is a generalization of Theorem 4.1 on stability to the present setting; the difficulty is the quantitative description of the fact that the influence of the possible supplementary zero in the amplitude $\tilde{h}_-$ with respect to h_- is arbitrarily small if the errors around $o(t)$ are correspondingly reduced. Essentially, this comes about because the extra zero is very close to the cut and the values of the amplitudes h_-, $\tilde{h}_-$ on the cut in this region are themselves small. Thus, although the phase of h_- is very different from that of $\tilde{h}_-$, the absolute error is small.

We prove the following:

Theorem 6.1. *Let* $h_+(v)$, $h_-(v)$ *be a pair of functions in* $P_1(-v_{th}, v_{th})$ *and such that*: (i) $h_+(v) \neq 0$ for $|v| > v_{th}$, $v \in \mathbf{R}$, *but* $h_-(v)$ *may vanish at most once at* v_0, $|v_0| > \bar{v}(t)$; (ii) $h_+(v)$ *has* n_+ *zeros and* $h_-(v)$ n_- *or* $n_- + 1$ *zeros in the complex* v *plane*; (iii) *they lead to observables* $o + \Delta o$ *in* $\Delta O_{P1}(t)$; (iv) *the distance between the zeros of* $h_+(v)$ *and those of* $h_-(v)$ *is larger than* $d > 0$. *Then, for any* $\varepsilon > 0$, $\gamma > 0$, *both sufficiently small, there exists* $\delta(\varepsilon;\gamma)$ *so that, for any other pair of functions* $\tilde{h}_+(v)$, $\tilde{h}_-(v)$, *with the same properties* (i)–(iii), *generating observables* $o + \Delta\tilde{o}(t)$, $\Delta\tilde{o}(t) \in \Delta O_{P1}(t)$, *the condition* $\|\Delta o - \Delta\tilde{o}\| < \delta$ *implies* $\|(h_+ - \tilde{h}_+, h_- - \tilde{h}_-)\|_H < \varepsilon$.

To prove this, we take first some preparatory steps:

(i) Let $z_0 = \exp(i\Phi_0)$ and consider two disks d_ρ of radius ρ, centered at z_0, z_0^*. Then, for any $\varepsilon > 0, \rho > 0$, there exist $r_1(\varepsilon;\rho)$, $\Delta\Phi(\varepsilon;\rho) > 0$ such that, if $0 < r < r_1$, $|\Phi - \Phi_0| < \Delta\Phi$, the Blaschke factor $B_s(z(v))$ with $B_s(z=1) = 1$ and zeros at $(1-r)\exp(\pm i\Phi)$ obeys:

$$|1 - B_s(z(v))| < \varepsilon \tag{6.2}$$

for z in $|z| < 1$ outside the disks d_ρ.

Denoting by B_{s+}, B_{s-} the factors with zeros at $(1-r)\exp(\pm i\Phi)$, (6.2) follows from the (straightforward) inequality:

$$|1 - B_{s+}B_{s-}| < |e^{i\theta} + B_{s+}(z)| + |e^{-i\theta} + B_{s-}(z)| < 24\mu/\rho \tag{6.3}$$

valid if $\mu/\rho < 1/4$, $\rho < 2$, if z lies outside two disks of radius ρ, centered at z_0, z_0^* and the zeros of $B_{s+,-}$ inside two disks of radius μ, centered at z_0, z_0^*.

(ii) For definiteness, we assume that $\tilde{h}_-$ contains one more pair of zeros than h_- (the other possibilities are similar or simpler). Then, (A.4, A.6, A.8) show that $\tilde{h}_-$ differs from h_- by a factor

$$\pi_s(v) = B_{s-}(v)\exp\left[-\frac{(v_{th}^2 - v^2)(\bar{v}^2 - v^2)^{1/2}}{2\pi v}\right.$$

$$\cdot\int_{v_{th}^2}^{v^2}\frac{\operatorname{Im}\ln B_s(\sqrt{\eta'})B_s(-\sqrt{\eta'})}{(\eta' - v_{th}^2)(\eta' - v^2)(\bar{v}^2 - \eta')^{1/2}}\,d\eta$$

$$\left.\equiv B_{s-}(v)e_{s-}(v) \right] \tag{6.4}$$

and it is convenient to estimate $|\tilde{h}_- - h_-|$ through:

$$|h_-(v) - \tilde{h}_-(v)| < |\tilde{h}_-(v) - \pi_s(v)h_-(v)| + |\pi_s(v)h_-(v) - h_-(v)|. \tag{6.5}$$

(iii) for any $\varepsilon > 0$, $\gamma > 0$, $\rho > 0$, $\rho_0 > 0$ and sufficiently small, we may find $r, \Delta\Phi$, such that, if the two new zeros lie at $(1-r)\exp(\pm i\Phi)$ with $0 < r < r_1$, $|\Phi - \Phi_0| < \Delta\Phi$, then $|\pi_s(v) - 1| < \varepsilon$ in the domain F_γ, outside the images through $v(z)$ of two semidisks of radius ρ centered at $\exp(\pm i\Phi_0)$ and outside a circle of radius $|v| = \rho_0$ around the origin.

Indeed, (6.2) implies that $|\operatorname{Im}\ln B_s(z)|$ may be made as small as one wishes on the interval $(\theta(-\bar{v}), \theta(\bar{v}))$, $z(v) = \exp(i\theta(v))$, by letting the zeros get close enough to $|z| = 1$, around any Φ_0, outside this interval. Thus, $|e_{s-}(v) - 1|$ may be made smaller than $\varepsilon/2$ in F_γ minus a disk of radius ρ_0 around $v = 0$. Another term $\varepsilon/2$ is obtained from (6.2), by possibly choosing the zeros even closer to the boundary. This justifies (iii).

(iv) Let m be a number, to be thought of as small, but independent of ε, and such that (s5) is true. Assume there are points $v, |v| > \bar{v}(t)$, $v \in \mathbf{R}$, where $|E_R(v)| \equiv |R(v)| < m$. If m is sufficiently small, there are just two intervals on the unit circle: (θ'_1, θ''_1), $(-\theta''_1, -\theta'_1)$ where this occurs, in view of (s3). Let further $(\theta', \theta'') \supset (\theta'_1, \theta''_1)$ be an interval where $|E_R(\theta)| < 2m$ (also unique in $(0, \pi)$). Then, if θ is outside $(\theta', \theta'') \cup (-\theta'', -\theta')$:

$$\left|\frac{\tilde{E}_R(\theta)}{E_R(\theta)} - 1\right| < \frac{\varepsilon_1}{2m} + \left[\varepsilon_1 C_1 + \sqrt{\varepsilon_1}\frac{C_2}{m}(\theta''_1 - \theta'_1) + C_3\frac{\varepsilon_1^{1/2\beta'}}{m}\right]. \tag{6.6}$$

Equation (6.6) is the analogon of (4.2): the first term is the bound on $|\Delta R|/|R|$; the bracket bounds $|\Delta\psi(\theta)|$. In the latter, the integral over φ is divided into two: one part over the whole circle minus the intervals (θ'_1, θ''_1), $(-\theta''_1, -\theta'_1)$ and its complement. The first part leads to the same bound as in (4.4). In the second part, let I be the interval, if it exists, where $|R(v)| < \sqrt{\varepsilon_1}$, with $\varepsilon_1 < m^2/4$; by (s5) and the conditions on ΔO_{R1}, $\operatorname{mes}(I) < C\varepsilon^{1/2\beta}$. Then, for ϕ in I, the contribution $\Delta\psi_c(\theta)$ to $\Delta\psi(\theta)$ is:

$$|\Delta\psi_c| < \frac{1}{2\pi}\int_I \frac{|\ln|(R+\Delta R)(\varphi)|| + |\ln|R(\varphi)||}{|\sin(\theta - \phi)|}\,d\phi + \frac{1}{\pi}\frac{|\Delta R(\theta)|}{|R(\theta)|}\int_I \frac{d\phi}{|\sin(\theta-\phi)|} < \frac{C_5}{m}\varepsilon^{1/2\beta'}$$

$$+ C_4\frac{\varepsilon_1}{2m} < \frac{C_3}{m}\varepsilon_1^{1/2\beta'}, \quad \beta' > \beta \tag{6.7}$$

where we have used the inequality: $\operatorname{mes}(\theta'_1, \theta') > m/\sup_\theta|dR/d\theta|$, a consequence of a mean value theorem and used (s5) to estimate the logarithms, since $|R + \Delta R| < m$. Finally, in the complement of I in (θ', θ''), it is true that $|\ln|1 + \Delta R(\phi)/R(\phi)|| < 2\sqrt{\varepsilon_1}$, so that we obtain the middle term in the brackets of (6.6).

(v) With (6.6), we can estimate as in (4.1) the difference $|\tilde{B}(\theta) - B(\theta)|$, for θ outside $(\theta', \theta'') \cup (-\theta'', -\theta')$:

$$|\tilde{B}(\theta) - B(\theta)| < \varepsilon_2(\varepsilon_1) \tag{6.8}$$

with the same notation as in (4.6). For $\theta \in (\theta', \theta'') \cup (-\theta'', \theta')$, we take simply:

$$|\tilde{B}(\theta) - B(\theta)| < 2. \tag{6.9}$$

With this, the following analogon of (4.6) is true (omitting the nucleon pole):

$$|B_+(z)\tilde{B}_-(z) - B_-(z)\tilde{B}_+(z)| < \varepsilon_2^{(1-\omega(z;m))}(2)^{\omega(z;m)} \tag{6.10}$$

where we have used Nevanlinna's inequality and $\omega(z;m)$ is the harmonic measure of the intervals $(\theta', \theta'') \cup (-\theta'', -\theta')$.

(vi) In (6.10), $\tilde{B}_-(z)$ contains one zero more than $B_-(z)$. We show now that, for ε_2 small enough, the supplementary zero of $\tilde{B}_-(z)$ lies in a domain D_{k0}^C bounded by the arcs $(\pm\theta', \pm\theta'')$ and the level line:

$$\varepsilon_2^{(1-\omega(z;m))}(2)^{\omega(z;m)} \equiv k_0 \tag{6.11}$$

where $k_0 > 0$ is independent of ε_2 and will be specified below. Since $0 < \omega(z;m) < 1$ in $|z| < 1$, the curve (6.11) consists of two arcs with endpoints at (θ', θ''), $(-\theta'', -\theta')$ in turn; as $\varepsilon_2 \to 0$, the area bounded by these arcs vanishes.

Let r_-, r_+ be the minimal distances of the zeros of B_-, B_+ to the boundary and consider the disk $|z| = 1 - r_+/2$. Let $\bar{\varepsilon}_2(\varepsilon_2; r_+)$ be the maximum of (6.10) over this disk. As $\varepsilon_2 \to 0$, it vanishes like a power of ε_2. We repeat now the reasoning of Theorem 4.1, (4.7–4.11) and conclude the existence of a domain Ω_+, which contains all the zeros of $B_+, \tilde{B}_+$ and made up of at most n circles, so that the sum of their radii does not exceed $2e\varepsilon_{3+}^{1/n_+}(\bar{\varepsilon}_2)$; we may choose $\bar{\varepsilon}_2$ so small that Ω_+ is completely contained in the disk $|z| = 1 - r_+/2$. Let then $r' = \min(r_-, r_+/2)$; if $k_0 = (r'/2)^{n_- + n_+}$, it is true that, on the circle $|z| = 1 - r'/2$,

$$|B_-(z)\tilde{B}_+(z)| > k_0, \tag{6.12}$$

Let now ε_2 be so small, that the line (6.11) lies outside the disk $|z| = 1 - r'/2$. Then, since $B_-, \tilde{B}_+$ do not vanish outside this disk in $|z| < 1$ and have unit modulus on $|z| = 1$, (6.12) is valid also along the closed curve made up of the complement in $|z| = 1$ of $(\pm\theta', \pm\theta'')$ and the arcs (6.11). It follows that, along this curve:

$$|B_+(z)\tilde{B}_-(z) - B_-(z)\tilde{B}_+(z)| < |B_-(z)\tilde{B}_+(z)|. \tag{6.13}$$

Rouché's theorem shows then that the number of zeros of $\tilde{B}_-(z)$ inside the domain bounded by this curve is the same as that of $B_-(z)$. This proves our point.

(vii) In Theorem 6.1, we do not require that $h_-(v,t)$ vanishes somewhere for $|v| \geqq \bar{v}(t)$ (of course, it may); if it does not, then $|h_-(v,t)| > m > 0$ and we may find $\varepsilon(m) > 0$ so that, if $\varepsilon < \varepsilon(m)$, the ratios $\tilde{R}(v) = \tilde{h}_-(v)/\tilde{h}_+(v)$, corresponding to amplitudes which differ in the norm (3.14) by less than ε from h_-, h_+, have the same variation of the phase around $|z(v)| = 1$ as $R(v) = h_-(v)/h_+(v)$. Thus, the freedom of allowing one more zero in $\tilde{h}_-(v)$ is not relevant. More accurately, the distance $\|(\tilde{h}_- - h_-, \ \tilde{h}_+ - h_+)\|_H$ has a lower nonzero bound $\varepsilon_B > 0$ if $\tilde{h}_-$ has $n_- + 1$ zeros, but may be made vanishingly small if only n_- zeros are used. However, it may be that ε_B is very small, much smaller than a realistic level of errors ε_0. Then, it is of interest to estimate $\delta(\varepsilon_0)$, using both n_- and $n_- + 1$ zeros.

Qualitatively, we expect ε_B to play a role if $|h_-(v,t)|$ is small somewhere, compared to the expected level of errors ε_0. In the following, we assume that ε_0 is such that there exist intervals $(v'_2, v''_2) \subset (v'_1, v''_1) \subset (v', v'')$ so that $|\Phi(v)| \equiv |h_-(v)/E_S(v)| < \varepsilon_0/8$, $\varepsilon_0/4$ or $\varepsilon_0/2$ in turn inside them; outside them, $|\Phi(v)|$ is bounded from below by a positive constant, which we take equal to $\varepsilon_0/2$, for simplicity.

This ends the sequence of preparatory steps.

We now proceed to estimating $\delta(\varepsilon_0)$, so that, if $\|\Delta o\| < \delta(\varepsilon_0)$, $\|(h_+ - \tilde{h}_+, \ h_- - \tilde{h}_-)\|_H < k\varepsilon_0$ with $k = O(1)$. First, we notice that there exist two domains, both denoted by $d(\varepsilon_0)$ in $|z(v)| < 1$ and with part of their boundary on $(z(v'), \ z(v''))$ and its complex conjugate, so that $|\Phi(v)| < \varepsilon_0/2$, if $z(v) \in d(\varepsilon_0)$. Indeed, let $2C > \varepsilon_0$ be such that $|\Phi(v)| < C$, for $|v| > v_{th}$. Then, Nevanlinna's inequality for $\Phi(v)$ implies that $d(\varepsilon_0)$ includes the domain where the harmonic measure of the interval $(z(v'_1), \ z(v''_1))$ is larger than $(\ln(\varepsilon_0/2C)/(\ln(\varepsilon_0/4C)$. Consider now, for each $v \in (v', v'')$, the greatest disk $d(v){:}|z - z(v)| < r$ so that $d(v) \cap \{|z| < 1\}$ is contained in $d(\varepsilon_0)$. According to steps (i), (iii) above, for every $d(v)$, there exist values $r_1(v)$, $\Phi_1(v)$, such that, if the zeros of $B_s(v)$ lie in $d(z_s; v){:}\{z | z = (1-r)\exp(i\phi),\ |\phi - \theta(v)| < \phi_1, \ 0 < r < r_1\}$ and in the conjugate $d^*(z_s; v)$, then the inequalities:

$$|B_s(v) - 1| < \frac{\varepsilon_0}{4C}, \quad |e_S(v) - 1| < \frac{\varepsilon_0}{4C} \tag{6.14}$$

hold, the former in the cut v plane outside $d(v) \cup d^*(v)$, the latter in F_γ minus a disk $|v| < \rho_0$. With this, it is true that, if the zeros of $B_s(v)$ lie inside $D(z_s) \equiv \cup_v d(z_s; v)$ and $D^*(z_s)$, the inequality:

$$|\pi_s(v) h_-(v) - h_-(v)|/|E_S(v)| < 3\varepsilon_0/4 \tag{6.15}$$

holds for all v in F_γ with $|v| > \rho_0$. Indeed, outside $d(\varepsilon_0) \equiv \cup_v (d(v) \cup d^*(v))$, it is true that $|\pi_s(v) - 1| < \varepsilon_0/(2C)$ and $|\Phi(v)| < C$ from the theorem of the maximum modulus. Inside $d(\varepsilon_0)$, $|\Phi(v)| < \varepsilon_0/2$, $|B_s(v)| < 1$ and thus $|\pi_s(v) - 1| < 1 + \varepsilon_0/(4C) < 3/2$. This justifies (6.15).

Now, choose $m = \varepsilon_0/8$ in comments (iii)–(vi). According to comment (vi), for ε_2 small, (as described there) the zeros of $B_s(v)$ are contained in the domain $D^C_{k0} \equiv \mathrm{C} D_{k0} \cap \{|z| < 1\}$, with k_0 determined by the positions of the zeros of $h_-, h_+ (h_-$ has only n_- zeros). Since $(\theta', \theta'') \equiv (\theta(v'_2), \ \theta(v''_2)) \subset (\theta(v'), \ \theta(v''))$, it is true that, for ε_2 small enough, $D^C_{k0} \subset D(z_s)$ of the foregoing paragraph. This is the first condition to which $\varepsilon_2(\varepsilon_0)$, and thus $\delta = \|\Delta o\|$ is subjected, in order to fulfill (6.15).

We now write $\tilde{h}_-(v) = \pi_s(v) \tilde{h}_{1-}(v)$ and shall estimate the difference $|\tilde{h}_{1-} - h(v)|/|E_S(v)|$; the factor $\pi_s(v)$ is bounded by $|\pi_s(v)| < 1 + \varepsilon_0/(4C)$ in $F_\gamma \cap |v| \geqq \rho_0$. This proceeds in a manner similar to Theorem 4.1: we have to place bounds on the three terms $T_i(v)$, $i = 1, 2, 3$ in the analogon of inequality (4.23).

Consider first $T_1(v) \equiv |B_-(v) - \tilde{B}^0_-(v)|$ where we have written $\tilde{B}_-(v) = \tilde{B}^0_-(v) B_s(v)$. We repeat the steps of (4.8–4.16), changing the unit disk with $|z| < 1 - r'/2$ and ε_2 with $\tilde{\varepsilon}_2 =$ the maximum of the rhs of (6.10) over its boundary; We allow for the factor $B_s(z)$ explicitly; on $|z| = 1 - r'/2$:

$$|B_+(z) \tilde{B}^0_-(z) B_s(z) - B_-(z) \tilde{B}_+(z)| < \tilde{\varepsilon}_2; \tag{6.16}$$

If we choose ε_2 so that D_{k0} lies outside $|z| < 1 - r'/2$, it is true that:

$$|B_s(z_{i-})| > \frac{1}{2^2}\left(\frac{r_-}{2}\right)^2 > \frac{1}{2^2}\left(\frac{r'}{4}\right)^2 \tag{6.17}$$

so that (4.11) is replaced by:

$$\left|\prod_i (z - \tilde{z}_{i-})\right| < 2^{n_- + 6} \frac{6^{n_+}}{d^{n_+}} \frac{\tilde{\varepsilon}_2}{r'^2} \equiv \bar{\varepsilon}_{3-}(\varepsilon_2). \tag{6.18}$$

We may choose then Ω_-, Ω_+ so that they are even contained in $|z| < 1 - 3r'/4$ (cf. comment (vi)), so that, on the boundary $\partial\Omega_-$ of Ω_-, $|B_s(z)| > r_-^2/2^6$. With this, we conclude as in Theorem 4.1 that, for all v in the cut v plane, including, the cuts:

$$|\tilde{B}^0_-(v) - B_-(v)| < 2\bar{\varepsilon}_{5-}(\varepsilon_2)/r \tag{6.19}$$

(we have ignored the trivial complication due to the nucleon term, cf. (4.16)). We may now choose ε_2, so that the right hand side is less than ε_0.

We now move on to $T_2(v) \equiv |\tilde{E}^1_-(v) - E^1_-(v)|/|E_S(v)|$. We take over the argument of Theorem 4.1 on this point and have to estimate the left hand side of (4.18). Now, $f_-(v)$ becomes small at some points; as in comment (iv), we may give a bound on $|\tilde{E}^1_-(v) E_c(v)/E^1_-(v) - 1|$ outside an interval (v', v''), inside which $|E^1_-(v)|/|E_S| < \varepsilon_0/2$. The inequalities in (s5) for $R(v)$ may be transferred to $f_-(v)$ since $f_+(v) > k > 0$ at those points where $f_-(v)$ may vanish. In (6.6), we replace then ε_1 by $\|\Delta o\|$ and choose the latter so that the right hand side is less than ε_0. Finally, inside (v', v''), we choose $\|\Delta o\|$ so that $|\tilde{E}^1_-(v)|/|E_S(v)| < \varepsilon_0$ and thus, on this interval:

$$|T_2(v)| < \frac{|\tilde{E}^1_-(v)|}{|E_S(v)|} + \frac{|E^1_-(v)|}{|E_S(v)|} < 3\varepsilon_0/2. \tag{6.20}$$

There is no change in the argument for $T_3(v)$.

With this, we have shown how to choose $\|\Delta o\|$ to ensure $|\tilde{h}_- - h_-|/|E_S| < k\varepsilon_0$ in $F_\gamma \cap (|v| > \rho_0)$. Since $(h_-(v) - \tilde{h}_-(v))/E_S(v)$ is holomorphic in $|v| < \rho_0$, this choice ensures the inequality in all of F_γ and ends the proof.

Notice, the argument in Theorem 5.1 is independent of whether the values of $|t|$ are small or not and thus, using Theorem 6.1 instead of 4.1, we obtain the statement of Theorem 5.2 without the restriction $t \in (t_2, 0)$ (however, $h_-(v_N) = 0$ is possible only for $|v_N| > \bar{v}$).

7 Conclusions

We have presented an analysis of the stability of the reconstruction of the πN scattering amplitudes from polarization and differential cross section data, under small displacements of the latter. The procedure of reconstruction uses isospin invariance and analyticity in the energy at fixed momentum transfer and may be summarized as follows: at each fixed t, at zero errors, the data determines, via the ratio $R(v)$, (2.9–2.10), a minimal number of zeros of the two amplitudes $h_+(v,t)$, $h_-(v,t)$ and their position in the complex v plane; knowing these zeros, the algebra of Appendix A produces two corresponding "minimal" amplitudes h_{+0}, h_{-0}. The ambiguity consists of possible supplementary zeros, the same in h_+, h_-; this corresponds to the possibility of multiplying both minimal amplitudes h_{+0}, h_{-0} by the same factors (in any number) $\pi_s(v)$, (6.4); all observables are left unchanged in this process.

In [2, 3], we give arguments that, for some interval $(t_1(o), 0)$, such factors are absent, if the amplitudes belong to the class $\mathscr{U}$, Sect. 2. Clearly, one can formally obtain "minimal" amplitudes for any interval of values of t; Theorem 3.1 gives a way to verify whether the functions so obtained are the correct amplitudes or not. Its proof uses assumption (r1) in the definition of the class $\mathscr{U}$.

Now, the minimal amplitudes h_{+0}, h_{-0} are stable under small displacements of the data, provided these changes are such that the number of zeros stays the same in both amplitudes, and that the zeros of h_{+0} in the complex v plane are sufficiently far apart from those of h_{-0}. This is intuitively clear, is expressed in a precise manner in Theorems 4.1 and 6.1 and disposes of one part of the problem of stability.

On the other hand, if errors are present, one may accommodate within any given error channel functions $R(v)$ with wildly varying minimal numbers n_+, n_- of zeros and poles. Only the difference $n_+ - n_-$ may be obtained from the variation of the phase and the problem is to obtain the correct numbers. They cannot be derived simply by continuity with respect to t, because noncoincident zeros may appear in h_+, h_- from infinity, as we move down in t. It is the merit of the restrictions imposed in the class $\mathscr{S}$, in fact of the experimental fact (s1), that this occurrence is forbidden as long as conditions (V) of Theorem 3.1 are satisfied – at first for zero errors only. In the class $\mathscr{S}$, the minimal number of zeros is obtained from the variation of the phase of $R(v)$, which is a quantity stable under small fluctuations; if (V) is fulfilled, the corresponding minimal amplitudes are also the correct ones. Theorems 5.1 and 5.2 show that these statements stay true even in the presence of errors and conclude the discussion of stability.

In principle, the analysis of this paper gives a way to estimate the errors of the reconstructed amplitudes, given sufficiently small errors of the data. One should stress, however, that these bounds are of purely theoretical interest – it is, in fact their existence that matters – and they are of no numerical relevance for practical phase shift analysis. The latter is a process of great complexity, especially in its contemporary form, which incorporates, e.g. in πN scattering, a larger number of constraints than was mentioned here – in particular the unitarity of the partial waves. It is the impressive achievement of the work described in [14, 15] that one is nowadays in the possession of amplitudes, reproducing all available data, and consistent with the fundamental theoretical requirements of unitarity and analyticity and with isospin invariance. The present work – together with that of [2, 3] – shows that, within acceptable assumptions, the phase of the amplitudes is not merely fixed, but is, in fact, overconstrained by these requirements.

Appendix A

The formulae for the reconstruction of amplitudes

If the zeros v_i in the complex v plane of, say, $h_+(v,t)$, are known, then one defines successively:

$$B_+(v) = \frac{1 - z(v_B)z(v)}{z(v) - z(v_B)} \prod_i \frac{z(v) - z(v_i)}{1 - z(v)z^*(v_i)} \tag{A.1}$$

with

$$z(v) = \frac{\sqrt{v + v_{th}} - \sqrt{v_{th} - v}}{\sqrt{v + v_{th}} + \sqrt{v_{th} - v}} \tag{A.2}$$

and $v_B = (2\mu^2 - t)/4m$;

$$\ln E_+^1(v) = \frac{\sqrt{v_{th}^2 - v^2}}{\pi} \left[\int_{-\infty}^{-\bar{v}} + \int_{\bar{v}}^{\infty} \right] \cdot \frac{\ln f_+(v')}{(v' - v)\sqrt{v'^2 - v_{th}^2}} dv' \tag{A.3}$$

where the root in front of the brackets is positive for $|v| < v_{th}$; for $v_{th} \leq v \leq \bar{v}(t)$:

$$K_L(v) = \ln \left[\frac{\bar{K}(v) + 4iF_1(v)}{2B_+(v)B_+^*(-v)E_+^1(v)E_+^{1*}(-v)} \right] \Big/ \sqrt{v_{th}^2 - v^2} \tag{A.4}$$

where the logarithm is computed by continuity from threshold. The possible ambiguities of $2n\pi i$ appearing if the argument of the logarithm has zeros (it cannot become infinite) are removed by the sum rule (A.9) below. Further,

$$L_S(v) \equiv \frac{1}{2\pi} \int_{v_{th}^2}^{\bar{v}^2} \frac{\operatorname{Im} K_L(v')}{\eta' - v^2} d\eta' \tag{A.5}$$

and

$$L_A(v) = \frac{\sqrt{(v^2 - v_{th}^2)(v^2 - \bar{v}(t)^2)}}{2\pi v} \cdot \int_{v_{th}^2}^{\bar{v}^2} \frac{\operatorname{Re} K_L(\eta')\sqrt{\eta'}\, d\eta'}{(\eta' - v^2)\sqrt{(v_{th}^2 - \eta')(\eta' - \bar{v}(t)^2)}}. \tag{A.6}$$

Now, if

$$L(v) \equiv L_S(v) + L_A(v) \tag{A.7}$$

then

$$h_+(v) \equiv B_+(v) E_+^1(v) \exp[\sqrt{v_{th}^2 - v^2}\, L(v)] \tag{A.8}$$

achieves the reconstruction of h_+. The analyticity of $L_A(v)$ in the cut v plane requires:

$$\int_{v_{th}^2}^{\bar{v}^2} \frac{\operatorname{Re} K_L(v') d\eta'}{\sqrt{\eta'}\sqrt{(\bar{v}(t)^2 - \eta')(\eta' - v_{th}^2)}} = 0. \tag{A.9}$$

We also have the condition that $h_+(v)$, as defined by (A.8) is continuous at $v = \bar{v}$; in particular:

$$\operatorname{Im} L(v) \to \frac{\ln |h_+(\pm \bar{v})|}{\sqrt{\bar{v}^2 - v^2}} \tag{A.10}$$

as $v \to \pm \bar{v}$. But, as $v \to \pm \bar{v}, |v| < |\bar{v}|$

$$\operatorname{Im} K_L(v) \to \frac{1}{\sqrt{\bar{v}^2 - v_{th}^2}} \ln \frac{|h_+(\bar{v})|\,|h_+(-\bar{v})|}{2}. \tag{A.11}$$

Using (A.7, A.5), (A.10) implies:

$$\operatorname{Im} L_A(\bar{v}) = \frac{1}{2\sqrt{\bar{v}^2 - v_{th}^2}} \ln \frac{|h_+(\bar{v})|}{|h_+(-\bar{v})|}. \tag{A.12}$$

We also require:

$$\frac{d}{dv} \operatorname{Im} L(v) \to \frac{d}{dv} \ln f_+(v = \pm \bar{v}) \tag{A.13}$$

as $v \to \pm \bar{v}$. Equations (A.10, A.13) are sufficient to ensure that the real part of $h_+(v)$ is Hölder continuous of any index less than unity at $v = \pm \bar{v}$. The continuity of the derivative itself is not guaranteed, unless, of course, $o \in \mathcal{M}(\mathcal{S})$. We do not need, however, this sort of continuity in the definition of the inverse operator $\mathcal{L}_t$ in Sect. 3.

Appendix B

On a dispersion integral

We consider $E_R(v)$, (3.6). It is true that:

$$\ln E_R(v) = \frac{\sqrt{v_{th}^2 - v^2}}{\pi} \left[\int_{-\infty}^{-v_{th}} + \int_{v_{th}}^{\infty} \right] \cdot \frac{\ln R(v')}{(v' - v)\sqrt{v'^2 - v_{th}^2}} dv'. \tag{B.1}$$

Since $B_+(v)$, $B_-(v)$ have limits and $R(v) \to 1$ as $|v| \to \infty$, $v \in \mathbf{R}$, it follows that $\ln E_R(v)$ tends to a limit as $|v| \to \infty$, $v \in \mathbf{R}$, and lying, say, above the cuts. Since $|E_R(v)| \to 1$, this limit is purely imaginary, say $i\Phi_0$. If we can show that $|\ln E_R(v)|$ is bounded in the whole cut v plane, then it follows from the Phragmén–Lindelöf theorem that $\ln E_R(v)$ tends to $i\Phi_0$ uniformly in all directions, $0 < \arg v < \pi$ (and to $-i\Phi_0$ in $-\pi < \arg v < 0$)

There is the following (superficial) difficulty in showing that (B.1) is bounded. In view of the inequality (3.7), the following bound on the two dispersion integrals in (B.1)—denoted by $I(v)$—is easily derived: $(v = |v| e^{i\theta})$

$$|I(v)| < \frac{\text{const}}{|v|} \ln \frac{|v|}{|\sin \theta|}. \tag{B.2}$$

We have to dispose of the factor $\ln|\sin\theta|$. Let M be an upper bound on $|\ln E_R(v)|$ on the real axis and consider a contour made up of a semicircle of radius R on which (B.2) holds and the interval $(-R, R)$. It is true that:

$$|v I(v)| < \oint \partial G/\partial n(v, v') b(v') dv' \tag{B.3}$$

where $b(v')$ contains the various bounds and $G(v, v')$ is the Green's function of the domain enclosed, with pole at v and vanishing on the boundary. But $G(v, v') < G_0(v, v')$ where G_0 is the Green's function of the disk of radius R: $(v = re^{i\psi}, v' = Re^{i\theta})$:

$$\frac{\partial G_0}{\partial n}(v, v') = \frac{1}{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \psi) + r^2}. \tag{B.4}$$

Thus, for $|v'| = R$, it is true that $\partial G/\partial n(v, v') < \partial G_0/\partial n(v, v')$. With (B.4), one verifies directly that the part of $b(v')$ containing $\ln|\sin\theta|$ gives rise to a constant in the bound (B.3). It follows that $|vI(v)|$ is bounded by a constant (M) uniformly in $0 < \arg v < \pi$. This entails the boundedness of $|\ln E_R(v)|$.

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