APPENDIX

On the Duffing equation at large forcing and damping

by I.S. Stefanescu

Abstract: We present a theorem derived by the author [9] concerning the asymptotic behavior of Duffing's equation at large damping and external forcing.

In general, the periodic solutions of the Duffing equation

$$x'' + 2Dx' + x^3 = A \cos t\ , \quad A,D > 0 \tag{A.1}$$

may be unambiguously continued to neighbouring values of A,D. There exist, however, curves $D=D_c(A)$ in the A,D–plane across which their number and their stability properties change. A survey of the behavior of the periodic solutions at these singularity lines has been given in sect.2.4 and results of numerical calculations were discussed and shown in figures in sect.6.2.

This picture is clearly too complicated for a global analytical description at intermediate values of A. Also, with one exception, the literature appears to contain no à priori results concerning the domains in the A,D–plane where eq.(A.1) possesses a unique, symmetric solution: using a result of Cartwright and Littleton, W.S. Loud [16] showed (see O'MALLEY) that if an harmonic term kx is added in eq.(A.1), then this equation admits a unique periodic solution at every fixed A, provided D is large enough, with essentially D>const(k) A.

In [9], the author considers the case when both A and D are large. It turns out that in this limit, eq.(A.1) simplifies so that on the one hand its solutions may be approximated in a controlled manner, but on the other hand it stays complicated enough to display a non-trivial bifurcation structure which is the asymptotic form of the picture found in numerical calculations at intermediate values of A. A precise description of the domain of uniqueness in the A,D–plane, for A large enough, is obtained as a corollary. To state the result of [9], let D=D(A) be a monotonically increasing function of A. It turns out that the behavior of the solution of eq.(A.1) depends markedly on the choice of D(A). The following is true [9].

Theorem: Let $D=D_o(A)$ with

$$D_o(A) = \frac{1}{12\pi}\ln(A) - \frac{1}{3\pi}\ln\ln(A) + C_o + O\left(\frac{\ln\ln(A)}{\ln(A)}\right) \qquad (A.2)$$

where C_o is a constant, numerically accessible (see below).

Then:

a) If A is large enough, eq.(A.1) admits of a **unique, symmetric solution** if $D>D_o(A)$.

b) The maxima D_T^p and D_{PF}^p of the turning point and pitchfork singularities (p=1,2,...) interlace and lie, for A large enough, on the curve $D=D_o(A)$.

c) The positions of the maxima of the singularity lines are equidistant in the variable $A^{1/3}$ with a spacing

$$\delta(A^{1/3}) = (A_{PF}^{p})^{1/3} - (A_T^{p})^{1/3} = (A_T^{p+1})^{1/3} - (A_{PF}^{p})^{1/3} = \frac{\pi}{I\sqrt{3}} \simeq 0.71$$

$$I = \int_{-\pi/2}^{\pi/2} |\sin t|^{1/3} dt \qquad (A.3)$$

d) The half Poincaré map, eq.(2.1.1),

$$p:\ x(-\pi),x'(-\pi) \rightarrow -x(0),-x'(0) \qquad (A.4)$$

is asymptotically equivalent to the (one dimensional) mapping of a circle into itself

$$\Pi:\ \chi \rightarrow \beta\cos(\chi+\Sigma)\ \ (\mathrm{mod}\ 2\pi) \qquad (A.5)$$

with β, Σ known functions of D and A (see below).

It is true that

$$\beta = \frac{\exp(-\pi D)}{(D/A^{1/4})^{1/3}} K \tag{A.6}$$

where K is to be obtained from some special solutions of eq.(A.1) and of the variational equation around them [9], in a numerical manner (more precisely: K=lim K(A,D) as A→∞, and this limit exists and is bounded [9]). Eq.(A.2) is simply the solution with respect to D of the condition

$$\beta(A;D) = 1. \tag{A.7}$$

It turns out that to leading order in A

$$\Sigma = A^{1/3} \int_{-\pi/2}^{\pi/2} \sqrt{3}\ |\sin t|^{1/3} dt \tag{A.8}$$

and eq.(A.3) corresponds to $\delta(\Sigma) = \pi$.

It is a simple exercise to show that the mapping (A.5) has a unique fixed point for $\beta<1$ and all Σ and that it possesses infinitely many turning point and pitchfork singularity curves in the β,Σ plane, with a period 2π in Σ.

It is worthwhile explaining in more detail the equivalence of **p**, eq.(A.4), with Π, eq.(A.5) for A→∞. First, it is easy to see that the periodic solutions of eq.(A.1) are, in some sense, close to $(A \cos t)^{1/3}$, at least away from $t= -\pi/2$ (cf. sect.3). Thus, for the study of periodicity, we concentrate on the departures v of x from $(A \cos t)^{1/3}$ (for $|t+\pi/2|$ not too small). With this, it turns out that the natural independent variables for the problem are

$$\Theta_L = \sqrt{3}\ A^{1/3} \int_{t}^{-\pi/2} (\cos t')^{1/3} dt',\ \Theta_R = \sqrt{3}\ A^{1/3} \int_{-\pi/2}^{t} (\cos t')^{1/3} dt' \tag{A.9}$$

(L, R for left and right). Writing then p as $(v, dv/d\Theta_L)$ $(t=-\pi) \rightarrow (-v, -dv/d\Theta_R)$ $(t=0)$, it is shown in [9] that the only invariant sets of p (those containing in particular the fixed points) lie, for large A, in a disk of radius

$$R \simeq \exp(-D\pi/2)\, A^{-1/8} \tag{A.10}$$

centered at the origin in the $(v, dv/d\Theta)$ $(t=-\pi)$ plane. Under the action of Duffing's eq.(A.1), this disk is mapped (by p) into some area of magnitude $R^2\exp(-2\pi D)$. If $D \sim \ln(A)$, this area is, for large A, contained in a thin annulus around a circle of radius R_0R, with R_0 numerically computable. The thickness of the annulus vanishes as $A \rightarrow \infty$. Thus, the fixed points of p and of its iterates are given in this limit by a mapping like (A.5) of a circle into itself.

The methods used in deriving the results of [9] are taken from boundary layer perturbation theory (O'MALLEY, SMITH) and from the averaging theory of oscillations (BOGOLJUBOV-MITROPOLSKI, SANDERS-VERHULST).

In [14], J.G. Byatt–Smith derived by somewhat different methods asymptotic expansions for the solution of eq.(A.1) in the limit $A \rightarrow \infty$ with D held constant. These are formally similar to those of [9]. Unfortunately, this limiting situation requires for its treatment extensive numerical computations.

Finally, we note that if D(A) increases more rapidly than a logarithm, several possibilities concerning the behavior of the unique, periodic solution as $A \rightarrow \infty$ may occur. We refer to [8,17] for a detailed discussion.

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