The Onset of Bifurcations in the Forced Duffing Equation with Damping

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Abstract

The paper presents a complete description of the periodic solutions of the Duffing equation:

$$\ddot{y} + 2\Delta\dot{y} + y^3 = \Gamma\cos t$$

for large values of the forcing Γ and of the damping Δ. It contains a proof that the equation admits of an infinite sequence of bifurcation curves in the $\Gamma - \Delta$ plane, alternately of the saddle-node and odd periodic - simply 2π-periodic type, whose maxima lie at large Γ along the line:

$$\Delta_c(\Gamma) = \frac{1}{12\pi}\ln\Gamma - \frac{1}{3\pi}\ln\ln\Gamma + C(\Gamma)$$

where $C(\Gamma)$ has a finite limit as $\Gamma \to \infty$. The positions of the maxima are interlaced in asymptotically equal intervals of $\Gamma^{1/3}$, with a spacing of 1.403 units. For $\Delta > \Delta_c(\Gamma)$, the Duffing equation admits of a unique periodic solution if Γ is high enough.

These results are obtained by showing that the half-period Poincaré map offered by the Duffing equation is asymptotically equivalent to a map of a circle into itself, according to:

$$\chi \Rightarrow \beta\cos(\chi + \Sigma)$$

where χ is an angular variable and β, Σ depend on Γ, Δ. The numerical constants appearing in the circle map and its corrections are determined in the limit $\Gamma \to \infty$ by a parameter-free boundary layer equation and its variation.

1. Introduction

This paper is concerned with the Duffing equation with external forcing and damping in its simplest form:

$$\ddot{y} + 2\Delta\dot{y} + y^3 = \Gamma\cos\bar{t} \tag{1.1}$$

This equation exhibits a large variety of periodic solutions, possibly with periods different from that of the driving force and whose number changes with the values of the parameters Δ, Γ. These solutions may be studied at low forcing by approximate analytic methods (as in the classical books of Hayashi [1964], Stoker [1950], Hagedorn [1978], Landau & Lifshitz [1960]) or numerically in larger domains of the parameters Γ,Δ. There exist well-known diagrams, due to Y.Ueda[1980], in the $\Gamma - \Delta$ plane of the boundaries of regions where (1.1) admits of a certain type of periodic solutions, e.g. with a given period $2\pi m/n$. As the damping is decreased, these plots become increasingly intricate [Ueda, 1980]. For $\Delta > 0.1$ a diagram similar to Fig.1 of Ueda [1980] extending to $\Gamma \approx 200$ may be found in Höhler [1993] (A similar diagram, somewhat less detailed, exists in Sato *et al.* [1983]). If, at every fixed Γ, Δ is increased,one reaches always a region where (1.1) admits of a unique 2π-periodic solution. If a 2π-periodic solution $y_P(\bar{t})$ of (1.1) is unique, then it is necessarily odd-periodic (i.e. its Fourier series contains only odd harmonics):indeed, $y_{P1}(\bar{t}) \equiv -y_P(\bar{t}+\pi)$ is also a solution of (1.1)and, since it is by assumption identical to $y_P(\bar{t})$, it follows that, for all $\bar{t}$:

$$y_P(\bar{t}) = -y_P(\bar{t}+\pi) \tag{1.2}$$

From the $\Gamma - \Delta$ diagrams of Ueda [1980](also of Höhler [1993]) one sees that, moving along a line of constant Δ (at an intermediate value, say $0.2 < \Delta < 0.5$) starting at $\Gamma = 0$, there is first a small Γ-interval where (1.1) admits of a unique (odd-)periodic solution $y_P(\bar{t})$; this solution is also *stable*: it is an attractor in a Poincaré plot of period π. At a certain value of $\Gamma = \Gamma^L_{SN}$ a *saddle-node* bifurcation occurs and two new solutions, both odd-periodic, appear, one of which is stable, the other unstable. The "‘earlier"’ stable solution $y_P(\bar{t})$ may be continued smoothly past Γ^L_{SN} but "‘annihilates"’ at a higher $\Gamma = \Gamma^R_{SN}$ with the unstable solution originating at Γ^L_{SN} in a *reverse saddle-node* bifurcation. The remaining stable odd periodic solution, now unique, may be continued in Γ up to a point Γ^L_F where it undergoes a *pitchfork* bifurcation of a special type: it becomes unstable when continued past Γ^L_F and two simply 2π-periodic (*not* odd-periodic) stable solutions appear for $\Gamma > \Gamma^L_F$; these latter disappear again at a higher Γ^R_F, if the damping Δ is high enough[1]; at smaller values of Δ one traverses first an interval where the two stable solutions above lose their stability and two further 4π-periodic solutions appear with a shorter life (in Γ); if Δ is small enough one traverses a whole sequence of bifurcations leading to an attracting chaotic motion, presented in Ueda [1980, 1979] in well-known pictures. At values of Γ larger than Γ^R_F one meets again an interval of uniqueness, up to the next saddle-node bifurcation: for Δ sufficiently large,the saddle-nodes and flip bifurcations interlace.

A detailed study of eq.(1.1) at values of the forcing between ca.850 and 1500 and a damping $\Delta = 0.25$ is the object of a paper by J.G.Byatt-Smith[1986]; see also Byatt-Smith [1987][2]. The description of bifurcations given above is complicated by the appearance of 6π-periodic solutions, which also generate islands of chaotic motion as one moves up in Γ. There are also windows in Γ, where no chaotic motion exists but, e.g. 12π -periodic solutions. A study of (1.1) by analogue methods up to $\Gamma \approx 2000$ is presented by F.N.H. Robinson[1989], who points out the periodicity in $\Gamma^{1/3}$ of the way periodic solutions multiply (and disappear). A recent beautifully illustrated description of the formidable intricacy of the bifurcation diagrams for equation (1.1) is given by C.Bonatto, J.A.C.Gallas and Y.Ueda [2008]. Numerical evidence suggests that the sequence of bifurcations e.g. at $\Delta = 0.2$[Parlitz & Lauterborn, 1985] or $\Delta = 0.3$[Höhler, 1993] is infinite; the positions of the maxima of the saddle-node and flip (pitchfork) bifurcation curves appear to be equidistant in the variable $\Gamma^{1/3}$ (see also Sato *et al.* [1983] for an early attempt to explain this regularity, quite different from the present one)

The question arises whether an analysis of eqn.(1.1) can explain these phenomena from first principles. Since the complexity of the diagrams increases with decreasing damping Δ, it is tempting to start such an attempt from the region of high Δ where the solution of (1.1) is unique and give an explanation for the appearance of bifurcations as the damping becomes smaller. To the knowledge of the author, there exist almost no published descriptions of the domain of uniqueness of the solutions of (1.1), with the exception of the result of W.S.Loud [1955] who shows that, if an harmonic term $+ky$ is present in (1.1), then (1.1) has a unique periodic solution at every fixed Γ, provided Δ is large enough (essentially $\Delta > const \times \Gamma$); it seems, however, that the method is not readily extendable to

[1] the index F on the values of Γ comes from "‘flip"’: the pitchfork bifurcation of the period 2π Poincaré map is in fact a *flip* bifurcation of the half period Poincaré map (see Sect.III and VIII)

[2] The asymptotic expansions for the solutions of (1.1) given in the papers by J.G.Byatt-Smith appear also in the present work (Sects.V,VI), although derived in a different manner

the situation $k = 0$ of (1.1). Since the present work is concerned with the regime of large forcing Γ, I refer to some unpublished internal reports, which show the qualitative behaviour of the unique periodic solutions at high damping [Höhler & Stefanescu, 1987] and establish their uniqueness [Stefanescu, 1989] for large enough Γ in a domain above a line $\Delta(\Gamma)$ for which:

$$\lim_{\Gamma\to\infty} \frac{\ln\Gamma}{\Delta(\Gamma)} = 0 \tag{1.3}$$

In a subsequent internal report [Stefanescu, 1990] it was shown that uniqueness is lost as the damping traverses a line $\Delta \approx \ln\Gamma$ and that a sequence of bifurcations alternately of saddle-node and pitchfork type develops for lower Δ (if $\Gamma$is large enough). This result was based on a controlled approximation by means of averaging methods of the (half-period) Poincaré map provided by the Duffing equation (1.1).The main conclusion was that asymptotically the Poincaré map is well represented by a circle map:

$$\chi \Rightarrow \beta\cos(\chi + \Sigma) \tag{1.4}$$

with β, Σ depending on Γ and Δ. The bifurcation structure of (1.4) is then easy to obtain. This report remained unpublished at that time. Its results were summarized by the present author in an Appendix to the work of G.Höhler [1993]. The present work is essentially a repetition of the contents of this report, including the correction of some calculational mistakes [3] , a more careful development of the arguments and the addition of some drawings.

It is one of the results of this paper (and of [Stefanescu, 1990])that the "‘tips"’ of the saddle-node and flip bifurcations, which reach up to the highest values of Δ at fixed Γ, lie asymptotically along a line (cf.eq.(8.1) below):

$$\Delta(\Gamma) = \frac{1}{12\pi}\ln\Gamma - \frac{1}{3\pi}\ln\ln\Gamma + .. \tag{1.5}$$

These maxima are predicted to be asymptotically equidistant in the variable $\Gamma^{1/3}$ with a spacing given in leading order by (cf. eq:(8.3) and Fig. 13):

$$\delta(\Gamma^{1/3}) = \Gamma^{1/3}_{F,p} - \Gamma^{1/3}_{SN,p} = \Gamma^{1/3}_{SN,p+1} - \Gamma^{1/3}_{F,p} \approx 1.403 \approx \frac{\pi}{\sqrt{3}\int_{-\pi/2}^{\pi/2}|\sin t|^{1/3}dt} \tag{1.6}$$

The problem of the description of the Poincaré map of (1.1) at large Γ was taken up again a little later by G.Eilenberger and K.Schmidt ([1992], [1998]). These authors also derive the circle map (1.4) as a limit of the Poincaré map using, however, a different approximation scheme. Since these are the only papers which treat (1.1) in a spirit related to that of the present work - with similar conclusions - I shall sketch in the last section a comparison of the two approaches.

We introduce next the notation used throughout this work. We change in (1.1) variables to:

$$x = \frac{y}{\Gamma^{1/3}}, \qquad t = \bar{t} - \frac{3\pi}{2}, \qquad \varepsilon = \frac{1}{\Gamma^{2/3}}, \qquad \mu = \frac{\Delta}{\Gamma^{2/3}} \tag{1.7}$$

so that it becomes:

$$\varepsilon\ddot{x} + 2\mu\dot{x} + x^3 = \sin t \tag{1.8}$$

and $\Gamma \to \infty$ means $\varepsilon \to 0$, i.e. the coefficient of the second derivative vanishes. The problem of discussing solutions of (1.8) for small ε is a matter of *singular perturbation theory*, as expounded in the books by R.E.O'Malley [1974] or D.R.Smith[1985]. the question is well known (for linear equations)in the semiclassical treatment of quantum mechanics - the $(J)WKB$ method. Eqn.(1.8) is the form of Duffing's equation used throughout this paper.

[3] without consequences for the conclusion

If one changes the time unit further to $\hat{t} = t/\sqrt{\varepsilon}$,(1.8) becomes:

$$\frac{d^2x}{d\hat{t}^2} + 2r\frac{dx}{d\hat{t}} + x^3 = \sin(\sqrt{\varepsilon}\hat{t}), \qquad r = \frac{\mu}{\sqrt{\varepsilon}} = \frac{\Delta}{\Gamma^{1/3}} \tag{1.9}$$

This is the limit of extremely slowly varying forcing at small damping if $\Delta/\Gamma^{1/3}$ vanishes as $\Gamma \to \infty$ and is an invitation to apply the adiabatic theorem of classical mechanics [Landau & Lifshitz, 1960],[Arnold, 1978]. Eqn.(1.9) is the starting point of the approximations to the Poincaré map of (1.1) developed by G.Eilenberger and K.Schmidt[1992]

We give next a summary of the behaviour of (1.8) at high damping; the unique solutions that are obtained are qualitatively different depending on the relative magnitude of Δ and Γ (or of μ and ε). Assume $\Delta = \Delta(\Gamma)$ is a monotonically increasing function of Γ, for $\Gamma \to \infty$ (large Γ).

If, as $\Gamma \to \infty$, $\mu = \Delta/\Gamma^{2/3} > \mu_0 > 0$, we change variables in (1.8) to:

$$z = \mu x, \qquad \bar{\varepsilon} = \frac{\varepsilon}{\mu}, \qquad \bar{\mu} = \frac{1}{\mu^3} \tag{1.10}$$

and obtain:

$$\bar{\varepsilon}\ddot{z} + 2\dot{z} + \bar{\mu}z^3 = \sin t \tag{1.11}$$

As $\varepsilon \to 0$,(1.11) reduces to:

$$2\dot{z} + \bar{\mu}z^3 = \sin t \tag{1.12}$$

It is easy to show that, if $\bar{\mu}$ is bounded, (1.12) admits of a unique periodic solution which can be improved by iteration of (1.11) to a periodic solution $z_P(t)$ of the latter; further, $z_P(t)$ is unique [Stefanescu, 1989].

However, if $\mu \to 0$ as $\varepsilon \to 0$, eqn.(1.8) reduces in this limit to:

$$x^3 = \sin t \tag{1.13}$$

with the solution:

$$x_{00}(t) = (\sin t)^{1/3} \tag{1.14}$$

Corrections to $x_{00}(t)$ cannot be obtained by iterating (1.8), since the derivatives of $x_{00}(t)$ at $t = 0$ are not finite. We expect nevertheless (1.14) to be a good approximation to periodic solutions of (1.8) away from $t = 0$. The departures of the solutions of (1.8) from (1.14) near $t = 0$ are obtained by a *boundary layer* analysis (the book by C.M.Bender and St.A.Orszag [1978]contains an excellent introduction to this subject - treated otherwise in detail in the reference manuals on singular perturbation theory[O'Malley, 1974],[Smith, 1985]).Let:

$$t = \mu^{3/5}\tau, \qquad x = \mu^{1/5}\tau \tag{1.15}$$

so that (1.8) becomes:

$$\frac{\varepsilon}{\mu^{8/5}}\frac{d^2\eta}{d\tau^2} + 2\frac{d\eta}{d\tau} + \eta^3 = \mu^{-3/5}\sin(\tau\mu^{3/5}) = \tau - \frac{\mu^{6/5}}{6}\tau^3 + ... \tag{1.16}$$

To zeroth order in $\mu^{6/5}$, we are interested in that solution of (1.16) which behaves like $\tau^{1/3}$ as $\tau \to \infty$, so that it *matches* $x_{00}(t)$. If $\varepsilon/\mu^{8/5} \to 0$ as $\varepsilon \to 0$ (i.e.$\Delta/\Gamma^{1/4} \to \infty$), this solution is obtained by improving iteratively the solution of:

$$2\frac{d\eta}{d\tau} + \eta^3 = \tau. \tag{1.17}$$

with the same boundary condition. A "‘conjunction"’ of this solution inside the boundary layer with (1.14) outside it can be improved to a unique (odd-)periodic solution of (1.8)([Stefanescu, 1989], [Höhler & Stefanescu, 1987]: see Section III of this work for related procedures).

If, however, $\Delta/\Gamma^{1/4} \to 0$ as $\varepsilon \to 0$, the appropriate boundary layer quantities are:

$$t = \varepsilon^{3/8}\tau, \qquad x = \varepsilon^{1/8}\eta, \qquad \gamma = \frac{\mu}{\varepsilon^{5/8}}(= \frac{\Delta}{\Gamma^{1/4}}) \tag{1.18}$$

in terms of which (1.8) becomes:

$$\frac{d^2\eta}{d\tau^2} + 2\gamma\frac{d\eta}{d\tau} + \eta^3 = \varepsilon^{-3/8}\sin(\varepsilon^{3/8}\tau) = \tau - \varepsilon^{3/4}\frac{\tau^3}{6} + ... \tag{1.19}$$

As $\varepsilon \to 0$, the solutions of (1.19) obeying $\eta(\tau) \approx \tau^{1/3}$ as $\tau \to -\infty$ are oscillatory as $\tau \to +\infty$ and are damped in a "'time"' $\tau \approx 1/\gamma \to \infty$.

We distinguish thus three regimes of (1.8) for large Γ: (i)$\mu > \mu_0 > 0$ as $\varepsilon \to 0$; (ii)$\mu < \mu_0$ and $\varepsilon/\mu^{8/5} < const$ as $\varepsilon \to 0$; (iii)$\mu \to 0$, $\gamma = \mu/\varepsilon^{5/8} < \gamma_0$ as $\varepsilon \to 0$. Situations (i),(ii) lead to unique periodic solutions of (1.8) for large Γ; for a proof see the internal report [Stefanescu, 1989].The transition to nonuniqueness occurs in region (iii)(cf. eqns.(1.3),(1.5)). We shall thus assume throughout the present work that:

$$\gamma = \frac{\Delta}{\Gamma^{1/4}} = \frac{\mu}{\varepsilon^{5/8}} < \gamma_0 \qquad as \qquad \varepsilon \to 0 \tag{1.20}$$

and use the notation of (1.18). In view of (1.5), we find it convenient to use instead of μ the variable:

$$\kappa = \frac{\mu}{\varepsilon \ln(\frac{1}{\varepsilon})} = \frac{3}{2}\frac{\Delta}{\ln\Gamma} \tag{1.21}$$

so that bifurcations occur when $\kappa = O(1)$ as $\varepsilon \to 0$.

The paper is organized as follows: in Section II, some general preparatory statements are established concerning the boundedness of the solutions of (1.8) and the manner in which they approach each other in time. Section III introduces the *inner* and *outer* expansions associated to (1.8); these are combined and improved to two special, nonoscillatory solutions $X_L(t), X_R(t)$ of (1.8), defined for $t < 0$(L), $t > 0$(R) in turn. These solutions [4] are taken as references for $t < 0$, $t > 0$ in turn and the Poincaré map $\mathbb{P}$ is defined in terms of the differences:

$$v_L(t) = x(t) - X_L(t), \qquad t < 0; \qquad\qquad v_R(t) = x(t) - X_R(t), \qquad t > 0 \tag{1.22}$$

In Section IV a precise bound is derived on the region $\mathbb{D}_0(\varepsilon)$ of phase space where invariant sets of the map $\mathbb{P}$ may exist. Sections V and VI establish controlled approximations for *quarter period* Poincaré maps $\mathbb{P}_L$, $\mathbb{P}_R$ relating sections at $t = -\pi/2$ and $t = 0$ ($\mathbb{P}_L$) and $t = 0$ and $t = \pi/2$ ($\mathbb{P}_R$) in turn. In Section VII the complete (*half-period*) Poincaré map $\mathbb{P}(\varepsilon, \kappa)$ is written down and its limiting form for $\varepsilon \to 0$, the circle map Π (1.4), is established. Finally Section VIII discusses the extent to which bifurcation properties of the circle map can be transferred to those of the complete mapping $\mathbb{P}(\varepsilon, \kappa)$ for small, nonvanishing ε. In particular the statements of the Abstract concerning the asymptotic distribution of the bifurcation lines are derived. The paper is closed with some general remarks and a short comparison with related papers of G.Eilenberger and K.Schmidt[1992; 1998].

2. General Preparation

2.1. *Eventual boundedness of motions*

Lemma 2.1 *There exists a rectangle:*

$$\mathbb{R} : |x(t)| < B_1, \qquad |\frac{dx}{dt}| < \frac{B_2}{\sqrt{\varepsilon}} \tag{2.1}$$

[4] they are called *creeping* solutions by G.Eilenberger and K.Schmidt[1992]

so that all solutions paths $(x(t), \dot{x}(t))$ of (1.8) eventually get inside it. The constants B_1, B_2 are independent of ε, μ if μ and ε/μ are sufficiently small.

Proof:The argument is inspired by and similar to the one due to T.Yoshizawa [1953c; 1953b; 1953a] and presented in the book of G.Sansone and R.Conti [1964] . We consider the Lyapunov-type function $L(p, x, t)$ given by: $(p = dx/dt)$

$$L(p, x, t) = E(p, x, t) + D(p, x) \tag{2.2}$$

$$E(p, x, t) = \varepsilon\frac{p^2}{2} + \frac{x^4}{4} - x\sin t \tag{2.3}$$

$$\begin{aligned} D(p, x) = 0 \qquad & if \qquad p > \max[(\frac{|x|}{\mu})^{1/2}, (\frac{A}{\mu})^{1/2}] \\ = \varepsilon(p - (\frac{x}{\mu})^{1/2}) \qquad & if \qquad |p| < (\frac{x}{\mu})^{1/2}, x > A \\ = -2\varepsilon(\frac{x}{\mu})^{1/2} \qquad & if \qquad p < -(\frac{x}{\mu})^{1/2}, x > A \\ = -2\varepsilon\frac{x}{A}(\frac{|x|}{\mu})^{1/2} \qquad & if \qquad p < -(\frac{A}{\mu})^{1/2}, |x| < A \\ = 2\varepsilon(\frac{|x|}{\mu})^{1/2} \qquad & if \qquad p < -(\frac{|x|}{\mu})^{1/2}, x < -A \\ = -\varepsilon(p - (\frac{|x|}{\mu})^{1/2}) \qquad & if \qquad |p| < (\frac{|x|}{\mu})^{1/2}, x < -A \end{aligned} \tag{2.4}$$

where A is a constant, which will be chosen appropriately in the following. The function $L(p, x, t)$ is continuous and piecewise differentiable. In opposition to the functions considered in Sansone & Conti [1964] the function L in (2.2) is time-dependent, however in a "'mild"' manner: it is 2π-periodic. The choice of $D(p, x)$ is a modification of the one used by G.E.H.Reuter[1951], also presented[5] in Sansone & Conti [1964, p.376,ch.VII,§3]. Differentiation of (2.2) and use of the Duffing equation (1.8) establishes that:

$$\begin{aligned} \frac{dL}{dt} = -2\mu p^2 - x\cos t + \frac{\partial D}{\partial x}p - \frac{1}{\varepsilon}\frac{\partial D}{\partial p}(2\mu p + x^3 - \sin t) < -\delta < 0, \\ if \qquad x, p \in \mathbf{C}R_0, \qquad R_0 = \{|x| < A, |p| < (\frac{A}{\mu})^{1/2}\} \end{aligned} \tag{2.5}$$

if, e.g. $A \leq 2$ and μ, ε are appropriately small (depending on A, e.g. for $A = 2$, $\mu < 1/2$, $\varepsilon/\mu < 1/2$). The quantity δ is independent of (x, p) in $\mathbf{C}R_0$. As a consequence of (2.5),for any trajectory $(x(t), p(t)$ which starts at $t = t_0$ in $\mathbf{C}R_0$ and for any finite interval Δt,

$$L(x(t + \Delta t), p(t + \Delta t), t + \Delta t) - L(x(t), p(t), t) < const < 0 \tag{2.6}$$

as long as the trajectory stays in $\mathbf{C}R_0$.

For any (x, y) outside R_0 we define two functions, related to (2.2):

$$\begin{aligned} L_M(x, p) = \varepsilon\frac{p^2}{2} + \frac{x^4}{4} + D(x, p) + |x| \\ L_m(x, p) = \varepsilon\frac{p^2}{2} + \frac{x^4}{4} + D(x, p) - |x| \end{aligned} \tag{2.7}$$

[5] Use of the function offered for a more general situation by G.E.H.Reuter turns out to be appropriate only if the damping Δ increases faster than $\Gamma^{1/3}$

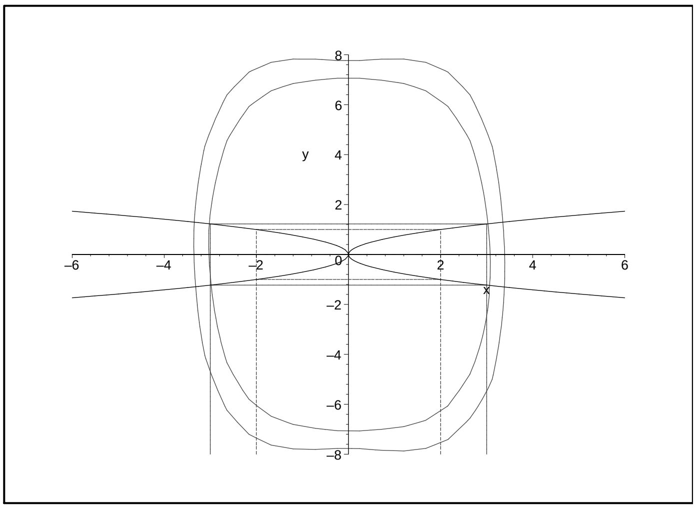

Figure 1: The rectangles R_0,R_1 used for $D(p,x)$ and the curves $\mathcal{C}_{lim}, \mathcal{C}_m$

which both increase indefinitely as $|x|$, $|p|$ tend to ∞ along any direction in the (x,p) plane, uniformly with respect to the direction. Further, we can replace the rectangle R_0 with a larger one R_1, so that both L_M, L_m are monotonically increasing outside R_1 along any ray in the (x,p) plane. One verifies that e.g. the rectangle

$$R_1 : \{|x| < A, |p| < \frac{1}{\sqrt{\varepsilon}}\} \tag{2.8}$$

fulfills this condition. The various domains of the $(x, \sqrt{\varepsilon}p)$ plane that appear in the definition of $D(p,x)$, eq.(2.4) are shown in Figure 1 for the situation $\varepsilon/\mu = 1/2$, $\varepsilon = 1/4$. The ε-dependence becomes weaker as $\varepsilon \to 0$ at constant ε/μ with this choice of variables, since the changes of $D(x, \sqrt{\varepsilon}p)$ between the various domains are proportional to $\sqrt{\varepsilon}$. The rectangle R_0 outside which $dL/dt < 0$ is shown with a dotted line, the "‘increased"’ rectangle R_1, eq.(2.8) with a continuous line. Let now

$$L_{lim} \equiv \max_{(x,p)\in\partial R_1} L_M(x,p) \tag{2.9}$$

where ∂R_1 denotes the boundary of R_1. By our choice of R_1,for a trajectory starting in $\mathbf{C}R_1$, as long as the corresponding function $L(x(t), p(t), t)$ assumes values strictly larger than L_{lim}, it is certain not to leave $\mathbf{C}R_1$. Indeed, by (2.9) crossing ∂R_1 requires $L \le L_{lim}$. But in $\mathbf{C}R_1$, the value $L(x(t), p(t), t)$ decreases monotonically with time by (2.6) so that, for any value $L_1 > L_{lim}$ there exists a time t at which $L(x(t), p(t), t) = L_1$.

Consider now such a value L_1 and the closed region $\bar{\mathcal{R}}$ in the (x,p) plane delimited by the closed curve:

$$\mathcal{C}_m : L_m(x,p) = L_1 \tag{2.10}$$

It contains the rectangle R_1 strictly in its interior. According to the above, all trajectories starting in $\mathbf{C}R_1$ reach at some time t the region $\bar{\mathcal{R}}$ since their corresponding L-function achieves the value L_1. Such a trajectory cannot leave the domain $\bar{\mathcal{R}}$ by traversing (or turning back from) the curve $\mathcal{C}_m$ because outside $\mathcal{C}_m$, $L > L_m(x,p) > L_1$ and this would contradict (2.6). Thus the trajectory is "‘trapped"’ in $\bar{\mathcal{R}}$. We obtain the statement of Lemma 2.1 by choosing $\mathbb{R}$ as a rectangle containing $\mathcal{C}_m$ in its interior: one verifies that this is so if $B_1 \approx 4$, $B_2 \approx 8$. This ends the proof of Lemma 2.1.

The curves C_{lim}, corresponding to $L_M(x,p) = L_{lim}$, eq.(2.9) C_m of (2.9)(enclosing C_{lim}) are shown in Fig. 1.

2.2. *The approach to some special solutions*

The following describes the manner in which a solution $x(t)$ of (1.8), trapped inside the rectangle $\mathbb{R}$ of Lemma 2.1, approaches a solution $x_0(t)$ also contained in $\mathbb{R}$ and subjected to the following supplementary

Condition 2.1 *There exist $a, b>0$, independent of ε, so that, for all t in some interval $[t_1, t_2]$ with $0 < t_1 < t_2 < \pi$ (mod π)*

$$|x_0(t,\varepsilon)| > a, \qquad |\frac{dx_0}{dt}(t,\varepsilon)| < b \tag{2.11}$$

Solutions of (1.8) contained in the rectangle $\mathbb{R}$ and obeying this condition will be shown to exist in Section 3. For any other $x(t)$ staying in $\mathbb{R}$ for $t > t_0$ we may state:

Lemma 2.2 *Assume $\mu/\varepsilon^{1/2} = \Delta/\Gamma^{1/3} < A_0$ and $\varepsilon/\mu = 1/\Delta \to 0$ as $\varepsilon \to 0$. Then, for ε sufficiently small, there exist constants K, C, independent of ε, so that, for any solution confined to the rectangle $\mathbb{R}$ of (2.1) and for $t \in [t_1, t_2]$:*

$$\max[|x(t) - x_0(t)|, \varepsilon^{1/2}|\frac{dx}{dt}(t) - \frac{dx_0}{dt}(t)|] < Ke^{-C\mu(t-t_1)/\varepsilon} \tag{2.12}$$

Proof: Let $C < 1$ and:

$$u(t) = (x(t) - x_0(t))e^{C\mu(t-t_1)/\varepsilon} \tag{2.13}$$

It verifies:

$$\varepsilon\frac{d^2u}{dt^2} + 2\mu(1-C)\frac{du}{dt} + u[3x_0(t)^2 + \frac{\mu^2(C^2-2C)}{\varepsilon}] + 3u^2x_0(t)e^{-C\mu(t-t_1)/\varepsilon} + u^3e^{-2C\mu(t-t_1)/\varepsilon} = 0 \tag{2.14}$$

Consider the Lyapunov function:

$$L_u = \frac{1}{2}(\varepsilon\frac{du}{dt} + 2\mu\bar{C}u)^2 + \varepsilon G(u,t) \tag{2.15}$$

with $\bar{C} \equiv 1 - C$ and:

$$G(u,t) = \int_0^u u\prime F(u\prime, t)du\prime \tag{2.16}$$

where $uF(u,t)$ denotes the last three terms of the l.h.s. of (2.14). The forms $F(u,t)$, $G(u,t)$ are positive definite for $t \in [t_1, t_2]$ if, e.g.$C < min(1/2, a^2/(4A_0^2))$. Using (2.14) we get:

$$\frac{dL_u}{dt} = -2\mu\bar{C}(u^2F(u,t) - \frac{\varepsilon}{2\mu\bar{C}}\frac{\partial G}{\partial t}) \equiv -2\mu\bar{C}u^2H(u,t) \tag{2.17}$$

One verifies that, with the choice of C above, if $\varepsilon/\mu = 1/\Delta$ is sufficiently small, $H(u,t)$ (of (2.17)) is positive definite for $t \in [t_1, t_2]$. Thus the solution paths $(u, du/dt)$ stay contained in the bounded domains defined at every $t \in [t_1, t_2]$ by:

$$L_u(t) < L_u(t_1) \tag{2.18}$$

But $L_u(t_1) = O(\varepsilon)$, since $(u, du/dt)$ are contained in the rectangle $\mathbb{R}$ of Lemma 2.1 and $\mu/\sqrt{\varepsilon} < A_0$. It follows from (2.18) and (2.15) that $G(u,t) = O(1)$ for $t \in [t_1, t_2]$ and since $G(u,t) = u^2F_1(u,t)$ with $F_1(u,t)$ strictly positive definite, it follows that $u(t) = O(1)$ for $t \in [t_1, t_2]$. Further, since $\varepsilon du/dt + 2\mu\bar{C}u = O(\varepsilon^{1/2})$, and $\mu/\sqrt{\varepsilon}$ is bounded, we conclude that $du/dt = O(\varepsilon^{-1/2})$ for $t \in [t_1, t_2]$. Returning now to (2.13) we obtain (using again the bound on $\mu/\sqrt{\varepsilon}$ for the evaluation of the time derivatives) the statement of the Lemma.

The bound $\mu/\sqrt{\varepsilon} < A_0$ delimits in the $\Gamma - \Delta$ plane a region where the damping may still be quite large to ensure at small ε uniqueness of the periodic solution of (1.8). In this paper the concern is the region of relatively small damping ($\Delta \approx const \ln\Gamma$) where uniqueness gets lost.The domain of larger damping may be treated completely, as is shown in the unpublished report [Stefanescu, 1989]

3. Inner and Outer Expansions. Reference Solutions

3.1. *Left hand reference solution*

The outer expansion

For small ε and μ and for t away from $n\pi$, we expect to obtain an approximate solution $x(t)$ of

$$\varepsilon\ddot{x} + 2\mu\dot{x} + x^3 = \sin t \tag{3.1}$$

by simply starting with:

$$x_{00}(t) = (\sin t)^{1/3} \tag{3.2}$$

and determining step by step with the help of (3.1) the coefficients $x_{kl}(t)$ of an expansion:

$$x_o(t) \equiv \sum_{k,l} \mu^k \varepsilon^l x_{kl}(t) \tag{3.3}$$

Equating to zero the coefficients of the various powers of μ and ε we obtain successively:

$$x_{10}(t) = -\frac{2\dot{x}_{00}}{3x_{00}^2}, \qquad x_{01}(t) = -\frac{\ddot{x}_{00}}{3x_{00}^2}, .. \tag{3.4}$$

and so on, so that we may state, in general:

Lemma 3.1 *With the definition in (3.3):*

$$x_{kl}(t) = t^{1/3-5k/3-8l/3} \sum_q a_{klq} t^{2q} \tag{3.5}$$

where the sum is uniformly and absolutely convergent for t in $[-\pi+\sigma, \pi-\sigma]$, for any $\sigma > 0$.

The proof is done by induction: the set of coefficients $\{x_{k0}(t)\}$ and $\{x_{0l}(t)\}$ form "'closed"' groups allowing the recursive determination of $x_{k0}(t)$ in terms of the $x_{k\prime 0}(t)$ with $k\prime < k$ and of $x_{0l}(t)$ in terms of $x_{0l\prime}$ with $l\prime < l$. The coefficient $x_{kl}(t)$ may be determined in terms of the $x_{k\prime l\prime}$ with $k\prime < k, l\prime \leq l$ and $k\prime \leq k, l\prime < l$. Thus we may determine successively the sets $\{x_{1l}\}$,$\{x_{2l}\}$,*etc.*. From (3.4), we see that,e.g.:

$$x_{10}(t) = -\frac{2}{9} t^{-4/3} \cos t \left(\frac{t}{\sin t}\right)^{4/3}, \qquad x_{01}(t) = \frac{2}{27} \frac{2+\sin^2 t}{t^{4/3}} (\frac{t}{\sin t})^{7/3}$$

The ratio $t/\sin t$ is a holomorphic, even and zero-free function of t in a disk of radius $\pi - \sigma$ around the origin; the same is true of $(t/\sin t)^{4/3}$ which justifies the statement about the convergence of the series in (3.5) in this situation. The same is true for $x_{01}(t)$ and is then transmitted recursively to all other coefficients. This ends the justification of Lemma 3.1.

The coefficients a_{klq} in (3.5) may be obtained step by step directly as follows: let $\alpha = \mu/t^{5/3}$,$\beta = \varepsilon/t^{8/3}$ and denote by:

$$\mathcal{D}(x) \equiv \varepsilon\ddot{x} + 2\mu\dot{x} + x^3 \tag{3.6}$$

Then one verifies that eq.(3.1) means(vgl.(3.5)) :

$$\mathcal{D}(t^{1/3} \sum_{k,l,q} a_{klq}\alpha^k\beta^l t^{2q}) = \sum_{k,l,q} b_{klq}\alpha^k\beta^l t^{2q+1} = \sum_q (-1)^q \frac{t^{2q+1}}{(2q+1)!} \tag{3.7}$$

where the b_{klq} are combinations of the $a_{k\prime l\prime q\prime}$ with $k\prime \leq k$, $l\prime \leq l$ and $q\prime \leq q$, but with only one term containing a_{klq} namely $3a_{000}^2 a_{klq}$. This equation allows the recurrent determination of the a_{klq} by equating the coefficients of $\alpha^k\beta^l t^{2q+1}$ on both sides, starting with $a_{000} = 1$. The coefficients with

$q = 0$ (or $k = 0$ or $l = 0$) build a "'closed"' group: the a_{kl0} may be determined successively from a_{000}. Then the calculation of a_{klq} for $q > 0$ requires the $a_{k\prime l\prime q\prime}$ with at least one strict inequality in the set $(k\prime \leq k, l\prime \leq l, q\prime \leq q)$.

An obvious question is: to what extent do we satisfy (3.1) if we restrict ourselves to a (K, L) truncation of (3.3)? From (3.7) we see that terms containing b_{klq} with $k > K$ or $l > L$ are, in general, nonvanishing, so that the action of $\mathcal{D}$ produces terms of $O(t\alpha^{K+1}, t\beta^{L+1})$,i.e. $O(t\mu^(K+1)/t^{5(K+1)/3} + t\varepsilon^(L+1)/t^{8(L+1)/3})$. This shows that a truncation of (3.3) is approximately a solution only for t away from zero, i.e. outside the boundary layer, which justifies the name *outer expansion*.

The inner expansion

The substitutions: $x = \varepsilon^{1/8}\eta$, $t = \varepsilon^{3/8}\tau$ change (3.1) to:

$$\frac{d^2\eta}{d\eta^2} + 2\gamma\frac{d\eta}{d\tau} + \eta^3 = \varepsilon^{-3/8}\sin(\varepsilon^{3/8}\tau) \tag{3.8}$$

which $\gamma = \mu/\varepsilon^{5/8}$. It is natural to look for a solution of (3.8) as an "'*inner expansion*"' in terms of the parameter $\varepsilon^{3/4}$:

$$\eta(\tau) = \eta_0(\tau) + \varepsilon^{3/4}\eta_1(\tau) + \varepsilon^{3/2}\eta_2(\tau) + ... \tag{3.9}$$

where the $\eta_q(\tau)$ are in turn solutions behaving like $\tau^{2q+1/3}$ as $\tau \to -\infty$ of the differential equations:

$$\frac{d^2\eta_0}{d\tau^2} + 2\gamma\frac{d\eta_0}{d\tau} + \eta_0(\tau)^3 = \tau \tag{3.10a}$$

$$\frac{d^2\eta_1}{d\tau^2} + 2\gamma\frac{d\eta_1}{d\tau} + 3\eta_0(\tau)^2\eta_1(\tau) = -\frac{\tau^3}{6} \tag{3.10b}$$

$$\frac{d^2\eta_2}{d\tau^2} + 2\gamma\frac{d\eta_2}{d\tau} + 3\eta_0(\tau)^2\eta_2(\tau) + 3\eta_1(\tau)^2\eta_0(\tau) = \frac{\tau^5}{5!}, etc. \tag{3.10c}$$

Because we wish the solution $\eta(\tau)$ to approach at large $-\tau$ the function $(\sin t)^{1/3}/\varepsilon^{1/8}$, it is natural to choose those solutions of (3.10a),(3.10b),etc. which behave like $\tau^{2q+1/3}$ as $-\tau \to \infty$, corresponding to the terms $(-1)^q\varepsilon^{3q/4}\tau^{2q+1/3}/(2q+1)!$ of the Taylor expansion. Concerning the expansion (3.9) it is easy to show

Lemma 3.2 *The solutions η_q appearing in eq.(3.9) exist and are uniquely determined by the requirement $\eta_q \approx \tau^{2q+1/3}$ as $\tau \to -\infty$. Their asymptotic expansion for $\tau \to -\infty$ is given by:*

$$\eta_q(\tau) = \tau^{2q+1/3}\sum_{k,l} a_{klq}\gamma^k\tau^{-5k/3-8l/3}, \qquad q = 0, 1, 2.. \tag{3.11}$$

with the same a_{klq} as in (3.5).

As before, one verifies by induction that (3.11) with coefficients $\tilde{a}_{klq}$ is a consistent asymptotic approximation for the solutions eqns.(3.10a),(3.10b),etc. obeying $\eta_q(\tau) \approx \tau^{2q+1/3}$ as $\tau \to -\infty$. The coefficients $\tilde{a}_{klq}$ are determined by substituting the asymptotic series (3.11) for each $\eta_q(\tau)$ in eq.(3.9) and requiring that (3.8) be verified to all orders in τ for all ε and γ. Denoting $t\prime \equiv \varepsilon^{3/8}\tau$, $\alpha\prime \equiv \gamma\tau^{-5/3}$, $\beta\prime \equiv \tau^{-8/3}$ and by $\tilde{\mathcal{D}}$ the action of the right hand side of eq.(3.8) on functions $\eta(\tau)$, one verifies:

$$\tilde{\mathcal{D}}(\sum_{klq}\tau^{1/3}\varepsilon^{3q/4}\tilde{a}_{klq}\gamma^k\tau^{-5k/3-8l/3+2q}) = \tau(\sum_{klq}\tilde{b}_{klq}(t\prime)^{2q}(\alpha\prime)^k(\beta\prime)^l) = \tau\sum\frac{(-1)^q(t\prime)^{2q}}{(2q+1)!} \tag{3.12}$$

with $\tilde{b}_{klq}$ analogous to b_{klq} of eq.(3.7). Eqn.(3.12) allows a recurrent determination of the $\tilde{a}_{klq}$ starting from $\tilde{a}_{000} = 1$. The equations determining from (3.12) the $\tilde{a}_{klq}$ are identical to the corresponding ones (3.7) for a_{klq} if we replace $t\prime \leftrightarrow t$, $\alpha\prime \leftrightarrow \alpha$, $\beta\prime \leftrightarrow \beta$. This shows that, indeed, $\tilde{a}_{klq} \equiv a_{klq}$.

We notice that the calculation of the asymptotic behaviour of η_q, $q > 0$ requires knowledge of the behaviours of $\eta_{q\prime}$ for $q\prime < q$, since $\eta_{q\prime}$ occurs in the equation for η_q (vgl.eq.(3.10c)). The relation between outer and inner expansions is apparent if we substitute $\tau = t\varepsilon^{-3/8}$, $\eta = x\varepsilon^{-1/8}$ in (3.5): one obtains the sum over q of the series (3.11), i.e. the asymptotic behaviour (at large $-\tau$) of the inner expansion is the same as the low t behaviour of the outer expansion.

We now show that, indeed, the requirement concerning the asymptotic behaviour selects unique solutions of (3.10a), (3.10b), (3.10c). We shall show directly that unique solutions exist admitting (3.11) as asymptotic expansion, from which the previous statement will follow. Using the method of the variation of parameters we set up an integral equation for the difference

$$u_q(\tau) \equiv \eta_q(\tau) - \eta_q(K, L) \tag{3.13}$$

where $\eta_q^{(K,L)}$ is the truncation of (3.11) after (K, L) terms. For $q = 0$ the equation reads:

$$u_0(\tau) = \int_{-\infty}^{\tau} e^{-\gamma(\tau-\tau\prime)}(v_1(\tau)v_2(\tau\prime) - v_2(\tau)v_1(\tau\prime))(H(\tau\prime) - 3\eta_0^{(K,L)}u(\tau\prime)^2 - u(\tau\prime)^3)d\tau\prime \tag{3.14}$$

where

$$H(\tau) \equiv -(\frac{d^2\eta_0^{(K,L)}}{d\tau^2} + 2\gamma\frac{d\eta_0^{(K,L)}}{d\tau} + (\eta_0^{(K,L)})^3 - \tau) = \gamma^K\tau^{-5K/3-8L/3}O(\gamma\tau^{-2/3}, \tau^{-5/3}) \tag{3.15}$$

and

$$v_{1,2}(\tau)e^{-\gamma\tau} \approx \frac{e^{-\gamma\tau}}{3^{1/4}|\tau|^{1/6}} \sin/\cos(\frac{3\sqrt{3}}{4}\tau^{4/3}) \tag{3.16}$$

are two linearly independent solutions of the equation:

$$\frac{d^2v}{d\tau^2} + 2\gamma\frac{dv}{d\tau} + 3(\eta_0^{(K,L)})^2v(\tau) = 0$$

The approximation indicated in eq.(3.16) is related to the WKB approximation and is discussed in a related context in Appendix C. One can show now by well-known methods that eq.(3.14) admits of a unique solution in a space of bounded continuous functions on $(-\infty, -a), a > 0$ with the norm $\sup|\tau^{5K/3+8L/3+1}u(\tau)|$. This solution is obtained by iteration of (3.14) which also sets a bound on the error made by truncating (3.11) at the (K, L) step: it is[6] is $O(\gamma^K\tau^{-5K/3-8L/3-1})$. Since the order of magnitude of the error is smaller than the last term included, this shows that the expansion (3.11) is indeed asymptotic; for $K, L = 0$ we obtain the statement of Lemma 3.2 for $q = 0$.

For $q > 0$, the equations (3.10a),(3.10b) and their analogues are linear and so are the corresponding integral equations (3.14); the solutions are given simply by the integral over (correspondingly modified) terms like $H(\tau)$ of (3.15); these terms contain solutions $\eta_{q\prime}$ with $q\prime < q$. The error of truncation after step (K, L) is now $C(q)\tau^{2q-1-5K/3-8L/3}\gamma^K$, with $C(q)$ a ($q-$dependent) constant.

The complete left hand reference solution

Although it is intuitively clear that eqns.(3.5) and (3.9) (using the expansion (3.11)) are expansions of the same solution of eqn.(3.1), it is not true that truncations at increasing K, L fulfill (3.1) increasingly well on the whole interval $[-\pi/2, 0]$. Following methods related to those of O'Malley [1974, ch.IV] and Smith [1985, ch.VI], we show how the two expansions have to be combined to yield a uniform approximation of the solution on $[-\pi/2, 0]$. We denote by:

$$x_o^{(K,L)} \equiv \sum_{k\leq K, l\leq L} \mu^k\varepsilon^l x_{kl}(t), \qquad x_i^{(Q)} \equiv \varepsilon^{1/8}\sum_{q\leq Q}\varepsilon^{3q/4}\eta_q(\tau) \tag{3.17}$$

[6] if $\gamma = 0$, then K must be set equal to 0

i.e. the $(K,L)-$, $Q-$ truncations of the sums in (3.5),(3.9) and take a number d, $0<d<3/8$. With these we set up a (K,L,Q)-approximant to a solution of (3.1):

$$x_a(t) \equiv \chi_o(t,\varepsilon^d)x_o^{(K,L)}(t) + \chi_i(t,\varepsilon^d)x_i^{(Q)}(t) \tag{3.18}$$

where $\chi_0(t,\varepsilon^d)$ is of class $\mathbb{C}^2$, $=0$ for $t>-a\varepsilon^d$ and $=1$ on $[-\pi/2,-b\varepsilon^d]$, $0<a<b$; further for $-\pi/2<t<0$,

$$\chi_i(t,\varepsilon^d) = 1-\chi_o(t,\varepsilon^d)$$

The function $x_a(t)$ is not a solution of (3.1) but is uniformly close for small μ and ε to such a solution on $-\pi/2<t<0$. Indeed, substitution of $x_o^{(K,L)}$ of (3.14) in (3.1) leaves terms of $O((\varepsilon/t^{8/3})^{L+1} + (\mu/t^{5/3})^{K+1})$ uncompensated; $x_i^{(Q)}$ verifies (3.1) up to terms of $O(\varepsilon^{3(Q+1)/4+3/8}\tau^{(2Q+3)})$;thus, using the notation of (3.6), for $-\pi/2<t<-a\varepsilon^d$ and $-b\varepsilon^d<t<0$ in turn:

$$\begin{aligned} &|\mathcal{D}(x_o^{(K,L)}) - \sin t| = O(\varepsilon^{(L+1)(1-8d/3)} + \mu^{(K+1)(1-5d/3)}), \\ &|\mathcal{D}(x_i^{(Q)}) - \sin t| = O(\varepsilon^{d(2Q+3)}) \end{aligned} \tag{3.19}$$

On the interval $(-b\varepsilon^d,-a\varepsilon^d)$ the functions χ_i,χ_o have derivatives of $O(\varepsilon^{-d})$ and second derivatives of $O(\varepsilon^{-2d})$. These are multiplied by the difference of the (truncated) asymptotic expansions of $x_o(t)$, $x_i(t)$ in this interval of t. As we have seen, the coefficients a_{klq} of these expansions are identical. The difference may then be estimated to be:

$$\begin{aligned} \Delta(Q,K,L,t) \equiv |(x_o-x_i)(t)| \le &\sum_{q>Q,k\le K,l\le L} a_{klq}t^{2q+1/3}\varepsilon^k\mu^l \\ &+\sum_{q\le Q}\varepsilon^{3q/4+1/8}u_q(\tau) \end{aligned}$$

where $u_q(\tau)$ are the "‘rest"’ functions introduced in (3.13). Letting $\tau=t/\varepsilon^{3/8}$ this expression is evaluated at $t=\varepsilon^d$ to be:

$$\Delta(Q,K,L,\varepsilon^d) \le C(K,L)\varepsilon^{(2Q+1/3)d} + C(Q)\varepsilon^{L(1-8d/3)}(\mu^K\varepsilon^{-5Kd/3})\varepsilon^{1/2-d}$$

where C(K,L),C(Q) are constants which depend on the truncation points, but not on ε. Similar estimates are valid for the first and second derivatives of the difference $\Delta(Q,K,L,t=\varepsilon^d)$. Clearly, it is possible to choose Q,K,L such that even $\Delta\varepsilon^{-2d}$ vanishes as $\varepsilon\to 0$ (so as to take into account the derivatives of the functions χ_i,χ_o in (3.18)) so that we may state, using the notation of (3.6):

Lemma 3.3 *The approximant x_a of (3.18) satisfies :*

$$\sup_{-\pi/2<t<0} |\mathcal{D}(x_a)-\sin t| < c_0(K,L,Q)\varepsilon^{c_1P} \tag{3.20}$$

where $P=\min(K,L,Q)$ and c_0,c_1 are positive constants.

With this we can now show the existence of a solution $X_L(t)$ of (3.1) which "‘interpolates"’ between the outer expansion (3.3) and the inner expansion (3.9): it is approximated by $x_a(t)$ uniformly on $-\pi/2<t<0$. We write:

$$X_L(t) = x_a(t) + r(t)$$

and require $X_L(-\pi/2) = x_a(-\pi/2)$, $\dot{X}_L(-\pi/2) = \dot{x_a}(-\pi/2)$. Then $r(t)$ is the solution of the integral equation :

$$r(t) = \frac{1}{\varepsilon}\int_{-\pi/2}^{t} \exp(-\frac{\mu}{\varepsilon}(t\prime-t))(v_1(t\prime)v_2(t)-v_1(t)v_2(t\prime))(\mathcal{D}-\sin t)(x_a)(t) + 3x_ar^2+r^3)dt\prime \tag{3.21}$$

where $v_{1,2}(t)\exp(-\mu/\varepsilon)(t+\pi/2)$ are solutions of the variational equation around $x_a(t)$ and $\mathcal{D}$ is the Duffing operator (3.6). We need here only rough bounds on these solutions. For $t < -\varepsilon^{3/8-d}$ (for some $0 < d < 3/8$) these solutions are well approximated by "‘WKB formulae"’ and may be chosen such that they have a limit as $\varepsilon \to 0$ (see Section 5.3 and Appendix C) for $t = O(\varepsilon^{3/8})$. It turns out that, if the solutions are chosen to be $O(1)$ at $t = 0$, then they are $O(\varepsilon^{1/16})$ at $t = -\pi/2$.The derivative with respect to τ is then O(1) at $t = 0$ but $O(\varepsilon^{-1/16})$ at $t = -\pi/2$. With this, using the bounds (3.20) one shows in a well-known manner that, if the integer P is sufficiently large, eqn.(3.21) admits of a unique solution of magnitude $\sup_{-\pi/2<t<0}|r(t)| < C\varepsilon^{c_1P-1}$, which can be obtained by iteration. We can thus conclude this section by:

Theorem 3.1 *Eqn.(3.1) admits of a solution $X_L(t;\varepsilon;K,L,Q)$ uniformly approximated to $O(\varepsilon^{c_1P-1})$ on $-\pi/2 < t < 0$ by $x_a(t)$,eq.(3.18) and which obeys: $X_L(-\pi/2) = x_a(-\pi/2)$, $dX_L/dt(-\pi/2) = dx_a/dt(-\pi/2)$.*

The estimates above are very rough and distort the numerical simplicity of the solution X_L: the dependence on K, L, Q is numerically irrelevant: for all practical purposes

$$X_L(t) = (\sin t)^{1/3}, -\pi/2 < t < -\varepsilon^d; \qquad = \varepsilon^{1/8}\eta_{00}(\tau), -\varepsilon^d < t < 0 \tag{3.22}$$

(with much freedom in the choice of $0 < d < 3/8$).The fact that the proof relies on the possibility to choose the integer P large originates from its ignoring the "‘destructive"’ action of the rapidly oscillating functions $v_{1,2}(t)$, which is apparent in their "‘WKB"’ form.It is this very weak dependence on the cutoff parameters (K, L, Q) (and thus on ε, μ) which justifies calling $X_L(t)$ the *(left hand) reference solution*. Its behavior near $t = 0$ is shown in Fig.2. The figure is virtually independent of ε,μ if time and magnitude are scaled appropriately.

3.2. *Right hand reference solution*

Choice of the inner expansion

We turn now to the interval $0 < t < \pi/2$. Obviously the outer expansion (3.3) is formally the same, with the same coefficients a_{klq}. Changing to the "‘inner"’ variables τ,η we consider again an expansion in terms of $\varepsilon^{3/4}$ similar to (3.9) and expect the asymptotic form of its various terms to reproduce the coefficients a_{klq}, appropriately rearranged. As will be apparent, the result is not the continuation to $\tau > 0$ of (3.9). There appears now an ambiguity in the definition of (3.9): the boundary condition that the solutions $\eta_q(\tau)$ of (3.10a),(3.10b),etc. should behave like $\tau^{2q+1/3}$ as $\tau \to \infty$ does not select a unique solution but is now obeyed by *all solutions* as a consequence of the damping term. The damping time is $1/\gamma \approx 1/(\varepsilon^{3/8}\ln 1/\varepsilon)$, which is shorter than the τ-duration of $O(1/\varepsilon^{3/8})$ of a quarter period. Equations (3.10a), (3.10b), etc. admit however of solutions with almost no oscillations even at times $\tau < 1/\gamma$: we write to this end, in (3.10a)(i.e.$q = 0$), for some integer r and appending an index R (for *right hand side*):

$$\eta_{0R}(\tau) = \sum_{k=0}^{r} \eta_{0kR}(\tau)\gamma^k + \gamma^{r+1}v(\tau) \equiv \eta_{0R}^{(r)} + u_0(\tau) \tag{3.23}$$

where the $\eta_{0kR}(\tau)$ $(k = 0, 1, ..)$ are, in turn, the solutions behaving like $\tau^{1/3-5k/3}$ as $\tau \to \infty$ of:

$$\frac{d^2\eta_{00R}}{d\tau^2} + \eta_{00R}^3 = \tau \tag{3.24a}$$

$$\frac{d^2\eta_{01R}}{d\tau^2} + 3\eta_{00R}^2\eta_{01R} = -2\frac{d\eta_{00R}}{d\tau}, ... \tag{3.24b}$$

$$\frac{d^2\eta_{02R}}{d\tau^2} + 3\eta_{00R}^2\eta_{02R} = -2\frac{d\eta_{01R}}{d\tau} - 3\eta_{00R}\eta_{01R}^2, ... \tag{3.24c}$$

These equations are obtained by equating the coefficients of various powers of γ after substituting (3.23) in (3.10a). Now the condition on the behaviour for $\tau \to \infty$ selects a unique solution because the damping term is absent[7]. It is easy to verify that the algorithm to obtain iteratively the asymptotic expansion of the solutions η_{0kR} leads indeed to:

$$\eta_{0kR} \approx \sum_l a_{kl0}\tau^{1/3-5k/3-8l/3} \tag{3.25}$$

with the a_{kl0} of (3.5). The function $u_0(\tau)$ is the solution of $O(\gamma^{r+1})$ of the equation:

$$\frac{d^2u_0}{d\tau^2} + 2\gamma\frac{du_0}{d\tau} + 3(\eta_0^{(r)})^2u_0 + 3\eta_0^{(r)}u_0^2 + u_0^3 = O(\gamma^{r+1}\tau^{1/3-5(r+1)/3}) \tag{3.26}$$

Such a solution may be obtained by iteration, repeating the argument of (3.14).

The equations for

$$\eta_{qR}(\tau) \equiv \eta_{q0R}(\tau) + \gamma\eta_{q1R}(\tau) + \gamma^2\eta_{q2R}(\tau) + ... \tag{3.27}$$

are linear and similar to (3.24b),(3.24c), deduced from (3.10b),(3.10c) expanding in powers of γ and setting appropriate boundary conditions at $\tau \to \infty$.

The complete right hand reference solution

We can now repeat the argument of section 3.1 and obtain a solution for $0 < t < \pi/2$ from a superposition like (3.18):

$$x_{aR}(t) \equiv \chi_{oR}(t, \varepsilon^d)x_{oR}^{(Q,K,L)}(t) + \chi_{iR}(t, \varepsilon^d)x_{iR}^{(Q,K)}(t) \tag{3.28}$$

where we have now appended an index R to the various terms,

$$x_{iR}^{(Q,K)} = \varepsilon^{1/8}(\eta_{0R}^{(K)}(\tau) + \varepsilon^{3q/4}\sum_{1\le q\le Q}\eta_{qR}^{(K)}(\tau)) \tag{3.29}$$

and $\eta_{qR}^{(K)}$ are the sums (3.27) restricted to K terms. Repeating the steps following eqn.(3.19) we state directly:

Lemma 3.4 *The approximant $x_{aR}(t)$ of (3.28) satisfies:*

$$\sup_{0<t<\pi/2} |\mathcal{D}(x_aR) - \sin t| = O(\varepsilon^{c_RP}) \tag{3.30}$$

with $P = \min(K, L, Q)$ and c_R is an (ε - independent) constant.

We would like now to repeat the procedure leading to the integral equation (3.21): there is, however, a difficulty because a direct analogy to (3.21) would require an integration *backwards* in time starting at $t = \pi/2$. The exponential term increases in this case indefinitely as $\varepsilon \to 0$ and precludes our setting bounds on $r(t)$. We have to start the integration at $t = 0$ and set :

$$X_R(t) = x_{aR}(t) + r(t), \qquad X_R(0) = x_{iR}(0), \qquad \frac{dX_R}{d\tau}(0) = \frac{dx_{iR}}{dt}(0) \tag{3.31}$$

The values $x_{iR}(0)$ are obtained from the solutions of the equations for η_{qkR} ,$q \le Q$, $k \le K$ (cf.(3.24a), (3.24b), (3.24c),...):

$$X_R(0) = \varepsilon^{1/8}\sum_{0\le q\le Q, k\le K}\varepsilon^{3q/4}\gamma^k\eta_{qkR}(0) \tag{3.32}$$

and similarly for $dX_R/d\tau(0)$.We conclude this section by stating in analogy to Theorem 3.1:

[7] we do not give an explicit proof of this, because the paper contains many similar arguments

Theorem 3.2 *Eqn.(3.1) admits of a solution $X_R(t;\varepsilon;K,L,Q)$ uniformly approximated to $O(\varepsilon^{c_R P})$ on $0 < t < \pi/2$ by $x_{aR}(t)$,eq.(3.28) and which obeys: $X_R(0) = x_{aR}(0)$, (eq.(3.32)) $dX_R/dt(0) = dx_{aR}/dt(0)$.*

The discontinuity at $t = 0$

We evaluate now $X_L(0)$, $dX_L(0)/d\tau$ using (3.10). The correction $r(t)$ obtained from (3.21) may be made as small as one wishes, by letting the cutoff integers K, L, Q be large enough. Then it is true that :

$$X_L(0) = \varepsilon^{1/8} \sum_{q \le Q} \varepsilon^{3q/4} \eta_q(0) \equiv \varepsilon^{1/8} \eta_L(0) \tag{3.33}$$

where the $\eta_q(\tau)$ are the uniquely defined solutions of (3.10a),(3.10b),(3.10c). These solutions may be expanded in powers of γ, similarly to (3.23) (we append from now on an index L for symmetry):

$$\eta_q(\tau) \equiv \eta_{qL}(\tau) = \sum_{k \le K} \gamma^k \eta_{qkL}(\tau) \tag{3.34}$$

where the η_{qkL} verify the same equations (3.24a),(3.24b),(3.24c) with a boundary condition (i.e. a prescribed asymptotic behaviour) at $\tau \to -\infty$ instead of $\tau \to \infty$. It is easy[8] to relate the solutions corresponding to these two boundary conditions, which interchange τ and $-\tau$. One verifies:

$$\begin{aligned} &\eta_{00L}(\tau) = -\eta_{00R}(-\tau), \qquad \eta_{01L}(\tau) = \eta_{01R}(-\tau) \\ &\eta_{02L}(\tau) = -\eta_{02R}(-\tau), ...\eta_{10L}(\tau) = -\eta_{10R}(-\tau) \end{aligned} \tag{3.35}$$

so that:

$$\Delta x_i(0) = \varepsilon^{1/8} \sum_{q,k} \varepsilon^{3q/4} \gamma^k \eta_{qkL}(0)(1 - (-1)^{k+1}) = 2\varepsilon^{1/8} \sum_{q,k=2p} \varepsilon^{3q/4} \gamma^k \eta_{qkL}(0) \tag{3.36}$$

and

$$\Delta \frac{dx_i}{dt}(0) = 2\varepsilon^{1/8} \sum_{q,k=2p+1} \varepsilon^{3q/4} \gamma^k \frac{d\eta_{qkL}}{d\tau}(0) \tag{3.37}$$

Since $\eta_{00L}(0) \neq 0$, it follows that $X_L(0) \neq X_R(0)$, thus the two *reference* solutions are *not* the continuation of each other. There is a jump of $O(\varepsilon^{1/8})$ at $t = 0$ in going from the one to the other. The derivatives have a smaller jump of $O(\varepsilon^{1/8}\gamma)$. We call these solutions *"'reference"' solutions* because the motions which we study consist of small oscillations around them. Eilenberger and Schmidt call them *creeping solutions*: the left hand solution $X_L(t)$ is the motion of a particle which stays at the bottom of the moving potential well $V(x) = x^4/4 - x \sin t$ for all times away from $t = 0$; near $t = 0$, the velocity of the minimum of the well at $x_m(t) = (\sin t)^{1/3}$ becomes unbounded and the particle cannot follow it: for $t > 0$ it will oscillate around the minimum with a larger amplitude (see Sect. 6.1). It behaves as if it had received a *kick* at $t = 0$[9]. In order that the particle follow the minimum of the potential for $t > 0$ away from zero it has to start at $t = 0$ from $(X_R(0), dX_R/dt(0))$: this is the *right hand creeping solution*: it will stay near the minimum until $t \approx 3\pi/2$. In Fig.2 we show the appearance of the reference solutions η_R, η_L near $t = 0$ ($\eta_{R,L} = \varepsilon^{-1/8} X_{R,L}$).

[8] invoking the uniqueness of the solutions

[9] However, there is no real "'kick"' and in my view the model proposed by the authors with a discontinuous force at $t = 0$ is not a correct description of the appearance of the circle map for the Duffing equation

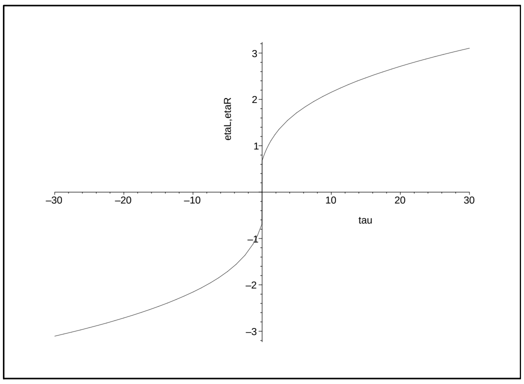

Figure 2: The solutions $\eta_{L,R}(\tau) = \varepsilon^{-1/8} X_{L,R}(t)$ near $\tau = 0$

3.3. *The Poincaré map*

We define the time-2π Poincaré map with respect to the reference solutions $X_L(t)$, $X_R(t)$ constructed in the previous subsections. If we choose the same cutoff parameters K, L in both, it is true that (to order ε^{cP}):

$$X_L(-\pi/2) = -X_R(\pi/2), \frac{dX_L}{dt}(-\pi/2) = -\frac{dX_R}{dt}(\pi/2)$$

As already noticed (cf.eq.(1.2)), if $x(t)$ is a solution of eqn(3.1), then $-x(t-\pi)$ is also a solution. Thus the solution starting at $t = \pi/2$ with the values $-X_L(-\pi/2)$, $-dX_L/dt(-\pi/2)$, is simply $-X_L(t-\pi)$, which we denote by $X_{L1}(t)$. This solution is a "‘reference"’ solution up to $t = \pi$ where it is replaced by $X_{R1}(t) \equiv -X_R(t-\pi)$. We write, for a solution $x(t)$ of (3.1):

$$\begin{aligned} &x(t) \equiv X_L(t) + v_L(t), -\pi/2 \leq t < 0, \qquad &&x(t) \equiv X_R(t) + v_R(t), 0 \leq t < \pi/2 \\ &x(t) \equiv X_{L1}(t) + v_{L1}(t), \pi/2 \leq t < \pi, \qquad &&x(t) \equiv X_{R1}(t) + v_{R1}(t), \pi \leq t < 3\pi/2 \\ &x(t) \equiv X_{L2}(t) + v_{L2}(t), 3\pi/2 \leq t < 2\pi \qquad &&, etc. \end{aligned} \tag{3.38}$$

We define the *time-2π Poincaré map* $\mathbb{P}_0$ by:

$$\mathbb{P}_0 : (v_L(-\pi/2), \frac{dv_L}{dt}(-\pi/2)) \Rightarrow (v_{L2}(3\pi/2), \frac{dv_{L2}}{dt}(3\pi/2)) \tag{3.39}$$

If $x(t)$ is the unique periodic solution of (3.1) then it must be odd periodic and leads thus also to a fixed point of the *half-period Poincaré map*:

$$\mathbb{P} : (v_L(-\pi/2), \frac{dv_L}{dt}(-\pi/2)) \Rightarrow (-v_{L1}(\pi/2), -\frac{dv_{L1}}{dt}(\pi/2)) \tag{3.40}$$

The symmetry $t \to t + \pi$, $x \to -x$ implies that, in fact:

$$\mathbb{P}_0 = \mathbb{P} \circ \mathbb{P} \tag{3.41}$$

Indeed, under $\mathbb{P}$ the point $P_i(x(-\pi/2), dx/dt(-\pi/2))$ corresponding to a solution $x(t)$, moves to $P_f(-x(\pi/2), -dx/dt(\pi/2))$; Also, the point $P_i^1(x_1(-\pi/2), dx_1/dt(-\pi/2))$ corresponding to the solution $x_1(t) \equiv -x(t+\pi)$ moves to $P_f^1(-x_1(\pi/2), -dx_1/dt(\pi/2))$. But $x_1(-\pi/2) = -x(\pi/2)$, $dx_1/dt(-\pi/2) = -dx/dt(\pi/2)$, i.e. $P_i^1 \equiv P_f$ and $-x_1(\pi/2) = x(3\pi/2)$, $-dx_1/dt(\pi/2) = dx/dt(3\pi/2)$. Therefore P_f^1 is at the same time the image of the original point P_i under $\mathbb{P} \circ \mathbb{P}$ and its image under $\mathbb{P}_0$. This shows the validity of (3.41). Now,the mismatch at $t = \pi/2$ between $-X_L(t-\pi/2)$ and $X_R(t)$ may be made arbitrarily small, simply by increasing the truncation orders, according to Theorems 3.1 and 3.2. We shall ignore this small correction in the following and regard $\mathbb{P}$ as being simply:

$$\mathbb{P}(\varepsilon, \mu) : (v_L(-\pi/2), dv_L/dt(-\pi/2)) \Rightarrow (v_R(\pi/2), dv_R/dt(\pi/2)) \tag{3.42}$$

In the next section, the definition will be further modified, by introducing another independent (time) variable.

4. The Invariant Sets of $\mathbb{P}(\varepsilon, \mu)$

4.1. *A characterisation of possible invariant sets*

The function $v_L(t)$, defined as the departure of a solution $x(t)$ from the reference function $X_L(t)$ (cf.eq.(3.38)) obeys:

$$\varepsilon \frac{d^2 v_L}{dt^2} + 2\mu \frac{dv_L}{dt} + 3X_L^2 v_L + 3X_L v_L^2 + v_L^3 = 0 \tag{4.1}$$

A similar equation holds for the function $v_R(t)$, eq.(3.38), the departure of $x(t)$ from the right hand side ($t > 0$) reference solution $X_R(t)$. It is convenient to introduce new independent variables for $t < 0$, $t > 0$, in turn:

$$\theta_L = 3^{1/2}\varepsilon^{-1/2} \int_t^{-\tau_0 \varepsilon^{3/8}} X_L(t\prime)dt\prime, \qquad \theta_R = 3^{1/2}\varepsilon^{-1/2} \int_{\tau_0 \varepsilon^{3/8}}^{t} X_R(t\prime)dt\prime \tag{4.2}$$

for a τ_0 such that $X_L(\tau)$, $X_R(\tau) \neq 0$ for $\tau < -\tau_0$, $\tau > \tau_0$, in turn. In this section we prove the following:

Theorem 4.1 *If the Poincaré map $\mathbb{P}(\varepsilon, \mu)$ has invariant sets, then, for ε sufficiently small, they are contained in a rectangle:*

$$\mathbb{D}_0 : \qquad |v_L(-\pi/2)|, \left| \frac{dv_L}{d\theta_L}(-\pi/2) \right| < M\varepsilon^{3/16+\kappa\pi/2} \tag{4.3}$$

with (cf.eq.(1.21))

$$\kappa = \kappa(\varepsilon) = \frac{\mu}{\varepsilon \ln(1/\varepsilon)} = \frac{3}{2} \frac{\Delta}{\ln \Gamma} \tag{4.4}$$

and M independent of ε .

With this definition of $\kappa(\varepsilon)$, the "‘damping factor"’ reads:

$$\exp(-\Delta t) = \exp(-\frac{\mu}{\varepsilon} t) = \varepsilon^{\kappa t} \tag{4.5}$$

Since (as will turn out) bifurcations occur when $\kappa = O(1)$, we shall use this latter notation from now on.

4.2. *A qualitative argument*

To explain intuitively the origin of this theorem, we notice first that, as a consequence of Lemma 2.2, *all* solutions of (4.1) must obey eventually

$$|v_L(-\pi/2)| < K\varepsilon^{C\kappa\pi/4} \qquad |dv_L/dt(-\pi/2)| < K\varepsilon^{-1/2}\varepsilon^{C\kappa\pi/4} \tag{4.6}$$

with K, C of (2.12). Indeed, the reference solution $X_L(t)$ obeys condition 2.1 of Sect. 2.2 on an interval $[-3\pi/4, -\pi/4]$: there $X_L(t) \approx (\sin t)^{1/3}$ (cf. eq.(3.22)) and one can choose $a, b \approx 2^{-1/6}$. Thus, invariant sets can be only subsets of the rectangle (4.6). Using the variable θ_L, eqn.(4.2), the last inequality is transformed into (with a redefinition of K):

$$|dv_L/d\theta_L(-\pi/2)| < K\varepsilon^{C\kappa\pi/4}.$$

With this in mind, we perform a change to a new dependent variable $w_L(\theta_L)$ through:

$$v_L(t) = \frac{w_L(\theta_L)}{(-X_L)^{1/2}} \varepsilon^{\kappa(t+\pi/2)} \varepsilon^{\alpha} \tag{4.7}$$

where $\alpha > 0$ is for the time being unspecified. Clearly, $v_L(-\pi/2) = w_L(\theta(-\pi/2))\varepsilon^{\alpha}$. The function w_L verifies for $t < 0$:

$$\frac{d^2 w_L}{d\theta_L^2} + w_L(1 + g_L(\theta_L)) - w_L^2 h(\theta_L) + \frac{h^2(\theta_L)}{3} w_L^3 = 0 \tag{4.8}$$

where:

$$h(\theta_L) = \frac{\varepsilon^{\kappa(t+\pi/2)} \varepsilon^{\alpha}}{(-X_L)^{3/2}} \tag{4.9}$$

$$g_L(\theta_L) = \frac{\varepsilon}{4} \frac{(dX_L/dt)^2}{X_L^4} - \frac{\varepsilon}{6} \frac{d^2 X_L/dt^2}{X_L^3} - \kappa^2 \varepsilon \frac{\ln^2(1/\varepsilon)}{3X_L^2} \tag{4.10}$$

Since $(-X_L)(t) \approx (\sin t)^{1/3}$ for $t < -\tau_0 \varepsilon^{3/8}$, one checks that $g_L(\theta_L) \approx \varepsilon(t^{-8/3} + t^{-2/3} \ln^2(1/\varepsilon))$ and is thus $O(1)$ at $t = O(\varepsilon^{3/8})$, drops off at larger $|t|$ and becomes $O(\varepsilon \ln^2(1/\varepsilon))$ at $t = -\pi/2$. The function $h(\theta_L)$ is not monotonical but has a minimum at a t of $O(1/(\kappa \ln \varepsilon))$ where it is $O(\varepsilon^{\kappa\pi/2+\alpha}(\kappa \ln(1/\varepsilon))^{1/2})$. It is $O(1)$ at $t = O(\varepsilon^{\kappa\pi+2\alpha})$, if $\kappa\pi + 2\alpha < 3/8$ but stays otherwise $o(1)$ down to $t = O(\varepsilon^{3/8})$. We denote:

$$p \equiv \min\left(\frac{\kappa\pi + 2\alpha}{3}, \frac{1}{8}\right) \tag{4.11}$$

so that $h(\theta_L)$ is $O(1)$ at $t \approx \varepsilon^{3p}$,if $p < 1/8$. The following is a qualitative argument for :

Statement 4.1 *If $p < 1/8$, the order of magnitude of $v_R(\pi/2)$ is $\varepsilon^{\alpha+\kappa\pi}$ (i.e. smaller by a factor $\varepsilon^{\kappa\pi}$ than $v_L(-\pi/2)$). If $p = 1/8$, then $v_R(\pi/2)$ is $O(\varepsilon^{3/16+\kappa\pi/2})$ and this order remains unchanged in the following half periods.*

The essential point in the argument is that the magnitude of the jump of the "‘reference"’ solutions $X_L(t)$, $X_R(t)$ is $O(\varepsilon^{1/8})$. Eqn.(4.8) describes an oscillatory motion in a time(θ)-dependent potential which has a single minimum at $w_L = 0$. If $|w_L| < const$ at $t = -\pi/2$, one expects that this motion remains bounded, uniformly with respect to ε, at least as long as $|h| < const$, i.e. down to times of $O(\varepsilon^{3p})$. At such values of t, according to (4.7), $v_L(t)$ is then of $O(\varepsilon^{p})$. The order of magnitude of $v_L(t)$ is likely to stay unchanged down to $t = 0$; there, the reference solution is changed from $X_L(t)$ to $X_R(t)$, which means a shift of $O(\varepsilon^{1/8})$. If $p < 1/8$. this shift is unnoticed: the function $v_R(t)$ has the same order of magnitude as $v_L(t)$ and we shall show that this is preserved up to t of $O(\varepsilon^{3p})$. Using a change of variables similar to (4.7):

$$v_R(t) = \frac{w_R(\theta_R)}{X_R^{1/2}} \varepsilon^{\kappa t + 3p/2} \tag{4.12}$$

we verify that $w_R(\theta_R)$ obeys:

$$\frac{d^2 w_R}{d\theta_R^2} + w_R(1 + g_R(\theta_R)) + w_R^2 k(\theta_R) + \frac{w_R^3}{3} k^2(\theta_R) = 0 \tag{4.13}$$

with $g_R(\theta : R)$ analogous to $g_L(\theta)$ of (4.6) and

$$k(\theta_R) = \frac{\varepsilon^{\kappa t + 3p/2}}{X_R^{3/2}(t)} \tag{4.14}$$

The function $k(\theta_R)$ is monotonically decreasing from $O(\varepsilon^{3p/2-3/16})$ at $t = O(\varepsilon^{3/8})$ to $O(\varepsilon^{\kappa\pi/2+3p/2})$ at $t = \pi/2$. Now, if at $t = O(\varepsilon^{3p})$, $v_R(t) = O(\varepsilon^p)$, it follows that $w_R(\theta_R)$ is $O(1)$ there and, since the motion described by (4.13) is oscillatory and - hopefully - bounded, it stays so up to $t = \pi/2$; there, consequently, $v_R(\pi/2) = O(\varepsilon^{\kappa\pi/2+3p/2})$. This is $O(\varepsilon^{\kappa\pi+\alpha})$ if $p < 1/8$. Thus, in this situation we start the next half period with a value of $v_L(-\pi/2+\pi)$ damped with respect to the original one by a factor $\varepsilon^{\kappa\pi}$. As announced in Statement 4.1, if $p < 1/8$, the new half cycle starts with a value of α, increased by $\kappa\pi$. After a finite number of cycles, α will be such that the inequality $p < 1/8$ is no longer valid. When this occurs, the magnitude of $v_L(t)$ at $t = 0$ is $o(\varepsilon^{1/8})$ and, since the jump of the reference solutions is $O(\varepsilon^{1/8})$, $v_R(t)$ is also of $O(\varepsilon^{1/8})$. Then, the change of variables (4.12) with $p = 1/8$ shows that $w_R(\theta)$ is $O(1)$ when t is of $O(\varepsilon^{3/8})$ and it follows that $v_R(\pi/2)$ is of $O(\varepsilon^{3p/2}) = O(\varepsilon^{\kappa\pi/2+3/16})$, as stated in Theorem 4.1.

To conclude, if we start with a value α such that $p < 1/8$, i.e.$\alpha < 3/16 - \kappa\pi/2$, it will increase in the following half cycles until it gets over this bound; in the succeeding half cycles it does not get any more below it. Indeed, if we start with $\alpha \geq 3/16 - \kappa\pi/2 \equiv \alpha_0$, $h(t)$ is $O(\varepsilon^s)$, with $s \geq 0$ at $(-t)$ of $O(\varepsilon^{3/8})$ and $v(t)$ is there of $O(\varepsilon^{1/8+s})$: it is thus at least a factor ε^s smaller than the jump of the reference solutions. The motion continues at $t > 0$ with oscillations of $O(\varepsilon^{1/8})$ around the reference $X_R(t)$; their amplitude decreases gradually (due to the factor $X_R^{-1/2}$) and, as a consequence of the damping, becomes $O(\varepsilon^{\kappa\pi/2+3/16})$ at $t = \pi/2$. Thus, once solutions are "‘trapped"’ in a rectangle (4.3), they stay there for all time.

For a proof of Theorem 4.1 (and of Statement 4.1), one has to place indeed bounds independent of ε on the magnitude of the solutions $w_{L,R}(\theta)$ of eqns.(4.8),(4.11) and also justify the preservation of the magnitude of the solutions in the transition region $(-\tau_0\varepsilon^p\ ,\tau_0\varepsilon^p)$.

4.3. *The interval* $-\pi/2 < t < -\tau_0\varepsilon^{3p}$

We show that, if the energy of the oscillations of w_L is bounded at $t = -\pi/2$, it stays bounded up to $t = -\tau_0\varepsilon^{3p}$, uniformly with respect to ε, if τ_0 is sufficiently large. To this end, we consider the energy associated with (4.8):

$$E(\theta) = \frac{1}{2}\left(\frac{dw_L}{d\theta_L}\right)^2 + \frac{1}{2}w_L^2\left[(1+g(\theta_L)) - \frac{2}{3}w_L h(\theta_L) + \frac{1}{6}w_L^2 h(\theta_L)^2\right] \tag{4.15}$$

The quantity in square brackets is positive definite and larger than $1/3 + g(\theta_L)$, so that $(g(\theta_L) > 0)$:

$$|w_L(\theta_L)| < \sqrt{6E(\theta_L)}, \qquad \left|\frac{dw_L}{d\theta_L}(\theta_L)\right| < \sqrt{2E(\theta_L)} \tag{4.16}$$

It follows that, if $E > 1$ and for those values of θ_L for which $|h(\theta_L)| < 1$

$$\left|\frac{dE}{d\theta_L}\right| = \left|\frac{1}{2}w_L^2\left[\frac{dg}{d\theta_L} - \frac{2}{3}w_L\frac{dh}{d\theta_L} + \frac{1}{3}w_L^2 h\frac{dh}{d\theta_L}\right]\right| < 3E(\theta_L)^2\left(\left|\frac{dg}{d\theta_L}\right| + 6\left|\frac{dh}{d\theta_L}\right|\right) \tag{4.17}$$

This inequality implies that, if the energy at $-\pi/2$ is bounded by a number E_0 and if $\tau_{0L}(E_0)$ is such that $h(\tau_{0L}\varepsilon^{3p})$ and $g(\tau_{0L}\varepsilon^{3p})$ are so small that:

$$\frac{1}{E_0} > 3g(\tau_{0L}\varepsilon^{3p}) + 6h(\tau_{0L}\varepsilon^{3p})$$

then $E(\tau_{0L}\varepsilon^{3p})$ is bounded. These inequalities imply that $|v_L(\tau_{0L}\varepsilon^{3p})|$, $\left|dv_L/d\theta_L(\tau_{0L}\varepsilon^{3p})\right|$ are bounded by $C(E_0)\varepsilon^p$ at these values of t. The constant increases as E_0 increases.

4.4. *The interval* $(-\tau_{0L}\varepsilon^{3p}, \tau_{0R}\varepsilon^{3p})$

Changing variables to

$$t = \sigma\varepsilon^{1/2-p}, \qquad v_L = V\varepsilon^p \tag{4.18}$$

one transforms (4.1) to:

$$\frac{d^2V}{d\sigma^2} + 2\kappa\varepsilon^{1/2-p}\ln\left(\frac{1}{\varepsilon}\right)\frac{dV}{d\sigma} + 3(X_L^2\varepsilon^{-2p})V + 3(X_L\varepsilon^{-p})V^2 + V^3 = 0 \tag{4.19}$$

For t in $(-\tau_{0L}\varepsilon^{3p}, 0)$, it is true that $|X_L\varepsilon^{-p}| < \tau_{0L}^{1/3}$. Also, one verifies that, at $t = -\tau_{0L}\varepsilon^{3p}$, $|dV/d\sigma|$ is $O(1)$. Thus the energy associated to (4.19) is bounded by a constant at $t = -\tau_{0L}\varepsilon^{3p}$. Further, it is true that, since the potential function is bounded from below by $V^4/6$,

$$|V(\sigma)| < 2E^{1/4} \tag{4.20}$$

Then, assuming $E > 1$, we may bound

$$\frac{dE}{d\sigma} < 8E^{3/4}\left(\frac{3}{2}\left|\frac{d(X_L\varepsilon^{-p})^2}{d\sigma}\right| + \left|\frac{dX_L\varepsilon^{-p}}{d\sigma}\right|\right)$$

and, integrating this inequality from $-\tau_{0L}\varepsilon^{3p}$ to 0:[10]

$$E(0)^{1/4} < E(-\tau_{0L}\varepsilon^{3p})^{1/4} + 2\left(\frac{3}{2}\tau_{0L}^{2/3} + \tau_{0L}^{1/3}\right) \tag{4.21}$$

It follows that both $|V|$ and $|dV/d\sigma|$ are bounded at $t = 0$ and therefore:

$$|v_L(0)| < C\varepsilon^p, \qquad \left|\frac{dv_L}{d\sigma}(0)\right| < C\varepsilon^p \tag{4.22}$$

With our definition (4.11) of p, the departure $v_R(t)$ of $x(t) = X_L(t) + v_L(t)$ from $X_R(t)$ is also of $O(\varepsilon^p)$ and the same is true for $dv_R/d\sigma(0)$. With the change of variables (4.18), with v_L replaced by v_R, we obtain an equation identical to (4.19) with $X_R(t)$ instead of $X_L(t)$. The same argument as before shows that the energy associated to it is bounded at $t = \tau_{0R}\varepsilon^{3p}$, for some (at this stage arbitrary) choice of τ_{0R}. It follows that $|v_R(\tau_{0R}\varepsilon^{3p})|$,$|dv_R/d\theta_R(\tau_{0R}\varepsilon^{3p})|$ are bounded by $C(\tau_{0R})\varepsilon^p$. Clearly, the value of $C(\tau_{0R})$ increases with τ_{0R}.

4.5. *The interval* $(\varepsilon^{3p}, \varepsilon^q), q < 3p$

In this interval $\varepsilon^{\kappa t} \approx 1$ and we may write:

$$k(\theta_R) \approx \frac{1}{\tau_{0R}^{1/2}}\left(\frac{\theta_{0R}}{\theta_R}\right)^{3/8}, \qquad \theta_{0R} \approx \frac{3\sqrt{3}}{4}\varepsilon^{4p-1/2}\tau_{0R}^{4/3} \tag{4.23}$$

The range of values of θ_R corresponding to this interval may become arbitrarily large if ε is chosen sufficiently small. We define w_R through equation (4.12): it follows from the previous paragraph that $|w_R|$, $dw_R/d\theta_R$ are O(1) at $t = \varepsilon^{3p}$. We shall show that w_R and $dw_R/d\theta_R$ stay bounded in the whole interval $(\varepsilon^{3p}, \varepsilon^q)$. $w_R(\theta)$ obeys eqn. (4.13) and by analogy to (4.15), with the same notation for the energy, but the replacement of $h(\theta_L)$ with $-k(\theta_R)$ we may write:

$$\frac{dE}{d\theta_R} = \frac{1}{2}w_R^2\frac{dg_R}{d\theta_R} + \frac{w_R^3}{3}\frac{dk}{d\theta_R} + \frac{1}{6}w_R^4 k\frac{dk}{d\theta_R} \tag{4.24}$$

[10] if $p = 1/8$ there are also constant contributions of the upper limit t=0; they do not play a significant part

We cannot repeat the argument of paragraph 4.3 because, if we try to increase the value of θ_{0R} to ensure an inequality like (4.17) we increase at the same time the bound on the possible energies (see the end of paragraph 4.4). One seems to need a rather long detour.

In (4.23), $dg_R/d\theta_R$, $dk/d\theta_R$ are strictly negative, so that the only term which may change sign is the middle one with w_R^3. Thus:

$$\frac{dE}{d\theta_R} < \frac{1}{3}|w_R^3|\left|\frac{dk}{d\theta_R}\right| \tag{4.25}$$

We can use now a bound on w_R similar to (4.16) to derive the inequality[11]:

$$\frac{dE}{d\theta_R} < CE^{3/2}(\theta_R)\left|\frac{dk}{d\theta_R}\right| \tag{4.26}$$

Unfortunately, we cannot draw any conclusions about the boundedness of E for large θ_R diretly from (4.26) unless some other restriction on $E(\theta_R)$ holds. We shall show in Appendix A that, in fact, $E(\theta_R)$ increases for large θ_R at most like $\theta_R^{3/4-s}$ for some $s > 0$, i.e. for θ_R in this time interval:

$$E(\theta_R) < const\left(\frac{\theta_R}{\theta_{0R}}\right)^{3/4-s} \tag{4.27}$$

Using this in (4.26) we obtain:

$$\frac{dE}{d\theta_R} < C\left(\frac{\theta_R}{\theta_{0R}}\right)^{9/8-3s/2}\left(\frac{\theta_{0R}}{\theta_R}\right)^{3/8}\frac{1}{\theta_R} \tag{4.28}$$

which leads by integration to an improved bound on $E(\theta_R)$:

$$E(\theta_R) - E(\theta_{0R}) < C\left(\frac{\theta_R}{\theta_{0R}}\right)^{3/4-9s/4} \tag{4.29}$$

Using this bound again in (4.26), we can further improve (4.29) and after a finite number of such steps, the power of (θ_R/θ_{0R}) decreases enough so that we can state:

$$E(\theta_R) < const \tag{4.30}$$

for all θ_R in the time interval $(\varepsilon^{3p}, \varepsilon^q)$.

4.6. *The interval* $(\varepsilon^q, \pi/2)$

This time integration of (4.26) leads directly to the desired bound:

$$\frac{1}{\sqrt{E(\theta_R)}} > \frac{1}{\sqrt{E(\theta_R(\varepsilon^q))}} - 2C(k(\theta_R(\varepsilon^q)) - k(\theta_R)) \tag{4.31}$$

we can always choose ε so small that the right hand side be positive [12].This leads then to an upper bound on $E(\theta_R)$ and thus on $E(\pi/2)$, as announced.

[11] capital C denotes a constant which needs not be specified in more detail

[12] the value $E(\theta_R(\varepsilon^q))$ is independent of ε, if ε is sufficiently small, according to paragraph 4.5

4.7. *Summary*

The important point concerning the bounds which were established above is that they are independent of ε, provided only ε is sufficiently small. We can now review the qualitative argument for Theorem 4.1 given in paragraph 4.2, whose gaps have now been filled in. As we have seen *all* solutions reach at some time $n\pi/2$ the interior of a rectangle $\mathbb{D}$ of size $K\varepsilon^{C\kappa\pi/4}$ around the reference solution [13] for a certain constant K. If there are invariant sets, they are contained in this rectangle for all times $m\pi/2$,$m = -\infty..\infty$. If $C\kappa\pi/4 > 3/16 - \kappa\pi/2 \equiv \alpha_0$ then the argument of paragraph 4.2 shows that those solutions that are in $\mathbb{D}$ at, say, $t = -\pi/2$, are contained at $t = \pi/2$ in the rectangle $\mathbb{D}_0$ of size $M_1\varepsilon^{\kappa\pi/2+3/16}$, eq.(4.3) which proves Theorem 4.1 for this situation. If the inequality is not satisfied, we may find a constant $\bar{M}_1$ so that the solutions are contained at $t = \pi/2$ in a rectangle $\mathbb{D}_1$ of size $\bar{M}_1\varepsilon^{\kappa(\pi+C\pi/4)}$, as argued in paragraph 4.2. If $\kappa(C\pi/4+\pi) > \alpha_0$, then the solutions reach at $t = 3\pi/2$ the interior of $\mathbb{D}_0$ and the proof stops at this stage. If not, we find $\bar{M}_2$ and a rectangle $\mathbb{D}_2$ of size $\bar{M}_2\varepsilon^{\kappa(C\pi/4+2\pi)}$ so that the solutions are contained in it at $t = 3\pi/2$. After a finite number of steps , the bound α_0 will be overcome and this ends the proof of Theorem 4.1.

Independently of the existence of invariant sets, the arguments of this section lead to the following:

Corollary 4.1 *Consider the solutions $v_L(t)$ starting at $t = -\pi/2$ in the rectangle (4.3) of Theorem 4.1 and the corresponding functions $w_L(\theta)$, $w_R(\theta)$ defined in (4.7) and (4.12). There exists then a constant M, indepedent of ε if ε is sufficiently small, so that*

$$|w_{L,R}(\theta_{L,R})|, \qquad \left|\frac{dw_{L,R}}{d\theta_{L,R}}\right| < M \tag{4.32}$$

for $t(\theta_{L,R})$ in $(-\pi/2, -\tau_0\varepsilon^{3/8})$ and $(\tau_0\varepsilon^{3/8}\pi/2)$,in turn.

This remark is used in the next sections to justify the averaging procedures employed there (see Arnold [1978, §52]).

5. The Left Hand Side Poincaré Map

5.1. *The left hand quarter period map*

In this and the next section we derive approximations to the half period Poincaré map $\mathbb{P}_{\mathbb{L}}$ restricted to the rectangle (4.3) of **Theorem 4.1**. We consider first the quarter period Poincaré map:

$$\mathbb{P}_{\mathbb{L}} : (v_L(-\pi/2), \frac{dv_L}{d\theta_L}(-\pi/2)) \Rightarrow (v_L(0), \frac{dv_L}{d\tau}(0)) \tag{5.1}$$

where τ is the "'boundary layer"' time variable introduced in Section 1, eq.(1.18),θ_L is defined in eq.(4.2) and v_L is a solution of eqn.(4.1) for $-\pi/2 < t < 0$. **Note:***in this section we shall drop the index "'L"' on the variable θ because θ_R, eq.(4.2),used for $t > 0$, does not appear at all. Also, for ease of notation, we write $g(\theta) \equiv g_L(\theta)$ of (4.10).* Instead of the rectangle eq.(4.3) we may consider a disk of the same magnitude and parametrize:

$$v_L(-\pi/2) = \varepsilon^{3/16+\kappa\pi/2}\Lambda\cos\Psi_0$$

$$\frac{dv_L}{d\theta}(-\pi/2) = -\varepsilon^{3/16+\kappa\pi/2}\Lambda\sin\Psi_0 \tag{5.2}$$

With the change of dependent variables (4.7), we are led to eq.(4.8) where now $\alpha = 3/16 + \kappa\pi/2$. The function $h(\theta)$ of (4.9) is $O(\varepsilon^{3/16+\kappa\pi/2})$ at $t = -\pi/2$ and $O(\varepsilon^{\kappa\pi})$ at $t = O(\varepsilon^{3/8})$. In an interval

[13] if the variable $\theta_{L,R}$ is used for the time

$I_L \equiv (-C_\varepsilon, -\tau_0\varepsilon^{3/8})$, where $C_\varepsilon \to 0$ when $\varepsilon \to 0$, we can set $\sin t \approx t$, so that eq.(4.2) implies $t \approx -\varepsilon^{3/8}(-\theta)^{3/4}$, and it follows that, for $|\tau|$ sufficiently large:

$$h(\theta) \equiv \frac{\varepsilon^{\kappa(t+\pi/2)}\varepsilon^{\kappa\pi/2+3/16}}{(-X_L)^{3/2}} \approx \varepsilon^{\kappa\pi}\left(\frac{-\theta_0}{-\theta}\right)^{3/8}, \qquad \theta_0 = -\frac{3^{3/2}}{4} \tag{5.3}$$

For[14] $\varepsilon^{3/8} = 0.003$, Fig.3 shows the appearance of $h(t)$; in the range of values of κ of interest (see below), the function $g(t)$ is much smaller than $h(t)$ (at $t = -10 \times \varepsilon^{3/8}$ it is $\approx 8.5 \times 10^{-5}$).

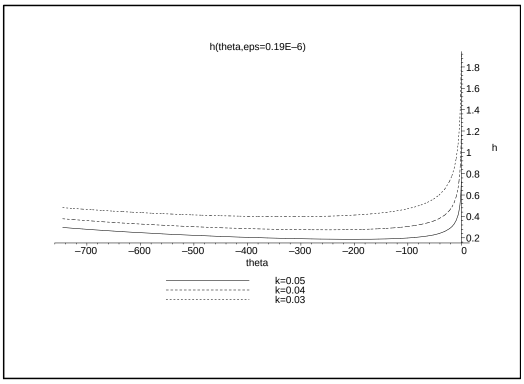

Figure 3: The function $h(\theta)$ for relevant values of the parameter κ

5.2. *The interval* $(-\pi/2, -\varepsilon^{3/8-\delta})$

The influence of the nonlinear terms of (4.8) can be analyzed due to the smallness of the function $h(\theta)$ in a manner inspired by and related to the averaging method of Bogolyubov and Mitropolsky[1961](especially chapter V of this reference). We use new dependent variables $R(\theta)$ and $\phi(\theta)$:

$$w_L(\theta) = R(\theta)\cos(\theta - \theta_0 + \phi(\theta)) \tag{5.4a}$$

$$\frac{dw_L}{d\theta} = -R(\theta)\sin(\theta - \theta_0 + \phi(\theta)) \tag{5.4b}$$

which transform (4.8) into the pair of equations:

$$\begin{aligned}\frac{dR}{d\theta} &= \frac{1}{2}g(\theta)R(\theta)\sin(2z) - \frac{1}{4}h(\theta)R(\theta)^2(\sin z + \sin(3z)) \\ &\quad + \frac{1}{12}h(\theta)^2R(\theta)^3(\sin(2z) + \frac{1}{2}\sin(4z))\end{aligned} \tag{5.5a}$$

$$\begin{aligned}\frac{d\phi}{d\theta} &= \frac{1}{2}g(\theta)(1 + \cos(2z)) - \frac{1}{4}h(\theta)R(\theta)(3\cos z + \cos(3z)) \\ &\quad + h(\theta)^2R(\theta)^2(\frac{3}{2} + 2\cos(2z) + \frac{1}{2}\cos(4z))\end{aligned} \tag{5.5b}$$

with

$$z(\theta) = \theta - \theta_0 + \phi(\theta) \tag{5.6}$$

According to section 4.1, the function $R(\theta)$ is bounded independently of ε for θ corresponding to t in an interval $(-\pi/2, -\tau_0\varepsilon^{3/8})$. Due to the smallness of $h(\theta)$ and $g(\theta)$, we expect both $R(\theta)$ and $\phi(\theta)$ to have a slow variation compared to θ; one might be tempted to replace eqs.(5.5a),(5.5b) with their

[14] The quantity $\varepsilon^{3/8}$ measures the width of the boundary layer; it corresponds to $\varepsilon = 0.187 \times 10^{-6}$

averages with respect to θ (or with respect to z). The fact that $h(\theta)$ is not integrable forbids however this simple averaging [15]. Following Bogolyubov & Mitropolski [1961], we perform a transformation of the dependent variables to new variables by:

$$R_1(\theta) = R(\theta) - \frac{1}{4}h(\theta)R(\theta)^2(\cos z + \frac{1}{3}\cos(3z)) \tag{5.7a}$$

$$\phi_1(\theta) = \phi(\theta) + \frac{1}{4}h(\theta)R(\theta)(3\sin z + \frac{1}{3}sin(3z)) \tag{5.7b}$$

The jacobian of this transformation is $1 + O(h)$ so that the transformation is *locally* invertible if ε and thus $h(\theta)$, are small enough. From the boundedness of $R(\theta)$ established in Section 4.1 it follows that $R_1(\theta)$ is also bounded, independently of ε. Moreover, one can show that the transformation (5.7a), (5.7b) is in fact invertible at fixed θ on its domain of values in the (R, ϕ) plane if $h(\theta)$ is sufficiently small. This question is discussed in Appendix B. The change of variables (5.7a),(5.7b) "'removes"' the terms of $O(h)$ in eqns.(5.5a),(5.5b): the equations for $R_1(\theta)$, $\phi_1(\theta)$ read:

$$\begin{aligned}\frac{dR_1(\theta)}{d\theta} &= \frac{1}{2}R(\theta)g(\theta)\sin(2z) + R(\theta)^3h(\theta)^2[\frac{19}{96}\sin(2z) + \frac{1}{12}\sin(4z) \\ &- \frac{1}{96}\sin(6z)] + O(h^3, hg, \frac{dh}{d\theta})\end{aligned} \tag{5.8a}$$

$$\begin{aligned}\frac{d\phi_1(\theta)}{d\theta} &= \frac{1}{2}g(\theta)(1 + \cos(2z)) + R(\theta)^2h(\theta)^2[-\frac{7}{24} - \frac{5}{16}\cos(2z) \\ &- \frac{1}{24}\cos(4z) - \frac{1}{48}\cos(6z)] + O(h^3, hg, \frac{dh}{d\theta})\end{aligned} \tag{5.8b}$$

In these equations it is understood that $R(\theta)$,$\phi(\theta)$ are replaced by functions of $R_1(\theta)$,$\phi_1(\theta)$ obtained by the inversion of eqns.(5.7a),(5.7b). To first order in $h(\theta)$ the latter reads:

$$R(\theta) = R_1(\theta) + \frac{1}{4}h(\theta)R_1(\theta)^2(\cos z_1 + \frac{1}{3}\cos(3z_1)) + O(h^2) \tag{5.9a}$$

$$\phi(\theta) = \phi_1(\theta) - \frac{1}{4}h(\theta)R_1(\theta)(3\sin z_1 + \frac{1}{3}\sin(3z_1)) + O(h^2) \tag{5.9b}$$

with

$$z_1 = \theta - \theta_0 + \phi_1(\theta) \tag{5.10}$$

Since $h(\theta)^2 \approx \varepsilon^{2\kappa\pi}(\theta_0/\theta)^{3/4}$ is again not integrable, one cannot draw directly conclusions about the behaviour of $R_1(\theta)$ and $\phi_1(\theta)$ over large intervals of θ. An attempt to remove the terms in $h(\theta)^2$ can achieve this only partially: the equation for $\phi_1(\theta)$ contains to orders $h(\theta)^2$ and $g(\theta)$ "'secular"' terms, i.e. terms which have nonzero average and which cannot be removed by a further transformation. It is relevant to notice that, in order to remove terms of higher order in eqns. (5.8a),(5.8b), one does not need to resort to the explicit inversion, eqns(5.9a),(5.9b), but regard $R(R_1, \phi_1)$, $\phi(R_1, \phi_1)$ as known functions of θ, whose derivatives are given by (5.5a),(5.5b). With this, we introduce new variables R_2, ϕ_2 by:

$$R_2 = R_1 + R(\theta)^3h(\theta)^2(\frac{19}{192}\cos(2z) + \frac{1}{48}\cos(4z) - \frac{1}{192}\cos(6z)) \tag{5.11a}$$

$$\phi_2 = \phi_1 + R(\theta)^2h(\theta)^2(\frac{5}{32}\sin(2z) + \frac{1}{96}\sin(4z) + \frac{1}{288}\sin 6z) \tag{5.11b}$$

As follows from Appendix B, this transformation is invertible under the same conditions as the one of eqns.(5.7a), (5.7b). Using (5.9a)(5.9b) for $R(\theta), \phi(\theta)$, it is true that:

$$R_1 = R_2 + O(h^2) \qquad \phi_1 = \phi_2 + O(h^2) \tag{5.12}$$

[15] as will be apparent the corrections may not be finite

Use of (5.5a),(5.5b) for $dR/d\theta$, $d\phi/d\theta$ and of (5.8a),(5.8b) for $dR_1/d\theta$, $d\phi_1/d\theta$ leads to:

$$\frac{dR_2(\theta)}{d\theta} = \frac{1}{2}g(\theta)R(\theta)\sin(2z) + O(h^3, \frac{dh}{d\theta}, gh) \tag{5.13a}$$

$$\frac{d\phi_2(\theta)}{d\theta} = -\frac{7}{24}R(\theta)^2h(\theta)^2 + \frac{g(\theta)}{2} + \frac{1}{2}g(\theta)\cos(2z) + O(h^3, \frac{dh}{d\theta}, gh) \tag{5.13b}$$

It is convenient to perform a further transformation:

$$R_3 = R_2 + \frac{1}{4}gR\cos(2z) \qquad \phi_3 = \phi_2 - \frac{1}{4}g\sin(2z) \tag{5.14}$$

which removes the "'nonsecular"' terms in $g(\theta)$ in (5.13a), (5.13b). This transformation brings additional terms in the equations corresponding to (5.13a),(5.13b), proportional to $dg/d\theta$ and $g(\theta)^2$. The former is dominant and, when its absolute value is integrated, leads to a term of $O(g(\theta))$. Now

$$\int_{\theta(-\pi/2)}^{\theta(-\tau_0\varepsilon^{3/8})} h(\theta)^3 d\theta = O(\varepsilon^{3\kappa\pi/2+1/16}/\ln(1/\varepsilon))$$

and vanishes as $\varepsilon \to 0$ so that we conclude from eqns.(5.13a),(5.13b),(5.14) that:

$$R_3(\theta) = R_{30} + O(\int_{\theta(-\pi/2)}^{\theta} h(\theta\prime)^3 d\theta\prime) + O(g(\theta)) \tag{5.15a}$$

$$\begin{aligned}\phi_3(\theta) = \phi_{30} - \frac{7}{24}\int_{\theta(-\pi/2)}^{\theta} R(\theta\prime)^2 h(\theta\prime)^2 d\theta\prime + \frac{1}{2}\int_{\theta(-\pi/2)}^{\theta} g(\theta\prime)d\theta\prime \\ + O(\int_{\theta(-\pi/2)}^{\theta} h(\theta\prime)^3 d\theta\prime) + O(g(\theta))\end{aligned} \tag{5.15b}$$

where R_{30}, ϕ_{30} are transformations of the initial conditions at $t = -\pi/2$ in (5.2). If, recalling (5.9a),(5.9b),(5.12) we invert the transformations (5.11a),(5.11b) and (5.7a), (5.7b) we obtain, using the notations in (5.2):

$$R(\theta) = \Lambda + O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta\prime)^3 d\theta\prime\right) + O(g(\theta)) + O(h(\theta)) \tag{5.16a}$$

$$\begin{aligned}\phi(\theta) = \Psi_0 - \frac{7}{24}\int_{\theta(-\pi/2)}^{\theta} R_{20}^2 h(\theta\prime)^2 d\theta\prime + \frac{1}{2}\int_{\theta(-\pi/2)}^{\theta} g(\theta\prime)d\theta\prime \\ + O\left(\int_{\theta(-\pi/2)}^{\theta} h(\theta\prime)^3 d\theta\prime\right) + O(g(\theta)) + O(h(\theta))\end{aligned} \tag{5.16b}$$

In eq.(5.16a) we see that, up to possible oscillations of $O(h(\theta))$, $R(\theta)$ stays constant at its initial value at $t = -\pi/2$ down to $t = O(\varepsilon^{3/8-\delta})$, where the second term containing $g(\theta)$ may become relevant. Evaluation of the integral in the first term of (5.16b) leads to:

$$\Phi_L \equiv -\frac{7}{24}\int_{\theta(-\pi/2)}^{\theta(-\tau_0\varepsilon^{3/8})} R_{20}^2 h(\theta)^2 d\theta = O(\frac{\varepsilon^{\kappa\pi}}{\gamma^{1/3}\kappa^{2/3}(\ln(\frac{1}{\varepsilon})^{2/3})}) \tag{5.17}$$

As it will become apparent, bifurcations occur when $\varepsilon^{\kappa\pi}/\gamma^{1/3} = O(1)$ so that this contribution to the phase due to nonlinear terms (these contain $h(\theta)$) is important: it decays indeed like $1/(\ln(1/\varepsilon))^{2/3}$, but this is very slow. The bounded function of κ and ε which multiplies the term under the $O()$ sign has a nontrivial behaviour and is shown in Fig.4;the magnitude of the phase variation implied by (5.16b) depends on the value of κ, i.e. of the damping: at $\varepsilon^{3/8} = 0.003$, for $\kappa = 0.04$ and $\Lambda = 1$ it is 0.46 rad, but at $\kappa = 0.02$ it measures 2.01 rad. The second term in (5.16b) is related to the linear part of (4.8) and brings a constant contribution at $t = -\tau_0\varepsilon^{3/8}$. We conclude this discussion by stating:

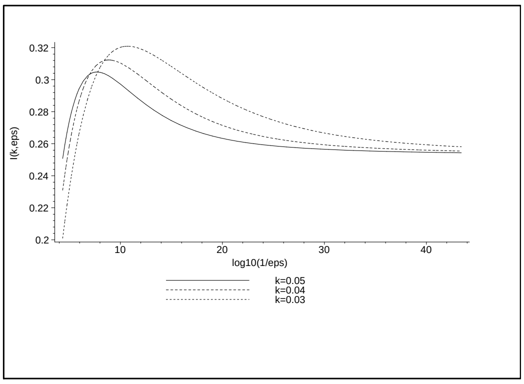

Figure 4: The dependence on ε of the factor multiplying the O() term in (5.17)

Lemma 5.1 *The solution of (5.4a),(5.4b) with the initial conditions* $R(-\pi/2) = \Lambda$, $\phi(-\pi/2) = \Psi_0$ *is given by eqns.(5.16a), (5.16b) for* t *in* $(-\pi/2, -\tau_0\varepsilon^{3/8})$.

In eq.(5.16a),(5.16b), the terms containing $g(\theta)$ originate in the linear part of eq.(4.8): we expect thus that if we solve the linear part of (4.8) (for a function $\tilde{w}_L(\theta)$):

$$\frac{d^2\tilde{w}_L(\theta)}{d\theta^2} + \tilde{w}_L(\theta)(1 + g(\theta)) = 0 \tag{5.18}$$

with the initial conditions:

$$\tilde{w}_L(\theta(-\pi/2)) = \Lambda\cos(\Psi_0 + \Phi_L), \qquad \frac{d\tilde{w}_L}{d\theta}(\theta(-\pi/2)) = -\Lambda\sin(\Psi_0 + \Phi_L) \tag{5.19}$$

with Φ_L of (5.17) we obtain for t close to $-\tau_0\varepsilon^{3/8}$ values which differ to $O(h)$ from those of the complete equation. With other words, the only effect of the nonlinear terms in (4.8) is the addition of a phase Φ_L to the simple harmonic evolution. For clarity, we do *not* expect that the solution of (5.18) with condition (5.19) approximates the solution of (4.8) over the *whole* interval $(-\pi/2, -\tau_0\varepsilon^{3/8})$; only the values near $t = -\tau_0\varepsilon^{3/8}$ are presumably well approximated. To render this precise, we analyze (5.18) using the same transformations (5.4a),(5.4b) to new variables - called $\tilde{R}, \tilde{\phi}$ - which obey equations like (5.5a),(5.5b) with $h(\theta) = 0$. Performing the change of variables (5.14) leads in analogy to (5.16a),(5.16b) to:

$$\tilde{R}_1(\theta) = \Lambda + O(g(\theta)), \qquad \tilde{\phi}_1(\theta) = \Psi_0 + \Phi_L + \frac{1}{2}\int_{\theta(-\pi/2)}^{\theta} g(\theta\prime)d\theta\prime + O(g(\theta)) \tag{5.20}$$

Comparison with (5.16a),(5.16b) shows that:

$$|R(\theta) - \tilde{R}(\theta)| = O(g(\theta)) + O(h(\theta)) + O(\int_{\theta(-\pi/2)}^{\theta} h(\theta\prime)^3 d\theta\prime) \tag{5.21a}$$

$$\begin{aligned} |\phi(\theta) - \tilde{\phi}(\theta)| &= O(\int_{\theta(-\pi/2)}^{\theta} h(\theta\prime)^3 d\theta\prime) + O(g(\theta)) + O(h(\theta)) \\ &+ \frac{7}{24}\int_{\theta}^{-\theta(\tau_0\varepsilon^{3/8})} \Lambda^2 h(\theta\prime)^2 d\theta\prime \end{aligned} \tag{5.21b}$$

If θ is sufficiently close to $\theta_0 \equiv \theta(-\tau_0\varepsilon^{3/8})$, e.g. corresponding to $t = -\varepsilon^{3/8-\delta}$ for δ sufficiently small, but nonzero, then all terms on the right hand side behave like positive powers of ε. As is easily verified, using (5.3) and (4.10), these are (up to logarithmic terms) in (5.21b): $\varepsilon^{3\kappa\pi/2+1/16}$, $\varepsilon^{8\delta/3}$, $\varepsilon^{\kappa\pi+\delta/2}$, $\varepsilon^{2\kappa\pi-\delta/3}$, in turn. There is some freedom in the choice of δ: a simple choice is $\delta = 3\kappa\pi/8$

where the $O(g(\theta))$ term is dominant and the differences (5.21a),(5.21b) are of $O(\varepsilon^{\kappa\pi})$ .If κ is large, this choice becomes inappropriate: if e.g. $\kappa\pi > 1$, the τ-time interval of $O(\varepsilon^{-\delta})$ is with this choice larger than the damping time $1/\gamma$. We are interested in τ-intervals slightly larger than the $O(1)$ scale but much smaller than $\varepsilon^{-3/8}$. We write (with some arbitrariness)

$$\delta = \frac{3r}{8} \qquad\qquad r = \min(\kappa\pi, \frac{1}{2}) \tag{5.22}$$

With this, for $\tau(\theta) < \varepsilon^{-\delta}$:

$$|w_L(\theta) - \tilde{w}_L(\theta)|, \qquad |\frac{dw_L(\theta)}{d\theta} - \frac{d\tilde{w}_L(\theta)}{d\theta}| = O(\varepsilon^r) \tag{5.23}$$

Now, the variational equation around the reference solution $X_L(t)$ to the Duffing equation (3.1) (or(1.8)) reads:

$$\varepsilon\frac{d^2\tilde{v}_L}{dt^2} + 2\mu\frac{d\tilde{v}_L}{dt} + 3X_L(t)^2\tilde{v}_L = 0 \tag{5.24}$$

With the same Liouville transformation (4.7) we may write for its solution :

$$\tilde{v}_L(t) = \varepsilon^{\kappa\pi/2+3/16}\frac{\tilde{w}_L(t)\varepsilon^{\kappa(t+\pi/2)}}{(-X_L(t))^{1/2}} \tag{5.25}$$

where $\tilde{w}_L(t(\theta))$ obeys (5.18) with initial conditons (5.19). Since $X_L(t) \approx -t^{1/3}$, one verifies from (5.25) that, for $t \approx \varepsilon^{3/8-\delta}$, both $v_L(t)$ and $\tilde{v}_L(t)$ are of $O(\varepsilon^{\kappa\pi+1/8+\delta/6})$. It is convenient to revert to the "‘time"’ τ and to variables $u_L(\tau)$ and $\tilde{u}_L(\tau)$ used in the boundary layer region with the scaling (1.18):

$$u_L(\tau) \equiv \varepsilon^{-1/8}v(t), \qquad \tilde{u}_L(\tau) \equiv \varepsilon^{-1/8}\tilde{v}_L(t), \qquad t = \varepsilon^{3/8}\tau \tag{5.26}$$

Since for small t, $\theta \approx -\tau^{4/3}$ both $du_L/d\tau$, $d\tilde{u}_L/d\tau$ are of $O(\varepsilon^{\kappa\pi-\delta/6})$ at $t = -\varepsilon^{3/8-\delta}$. Using (5.23), we conclude that, for such values of t ($\tau = \varepsilon^{-\delta}$):

$$|u_L(\tau) - \tilde{u}_L(\tau)| = O(\varepsilon^{\kappa\pi+r+\delta/6}), \qquad |\frac{du_L}{d\tau}(\tau) - \frac{d\tilde{u}_L}{d\tau}(\tau)| = O(\varepsilon^{\kappa\pi+r-\delta/6}) \tag{5.27}$$

The above is summarized in the statement of

Lemma 5.2 *The solutions v_L,eqn.(4.7), of eqn.(4.8) with the initial conditions (5.2) are approximated together with their derivatives at $t = -\varepsilon^{3/8-\delta}$ by the solutions of the variational equation (5.24) with the initial conditions:*

$$\begin{aligned}\tilde{v}_L(-\pi/2) &= \varepsilon^{3/16+\kappa\pi/2}\Lambda\cos(\Psi_0 + \Phi_L)\\ \frac{d\tilde{v}_L}{d\theta}(-\pi/2) &= -\varepsilon^{3/16+\kappa\pi/2}\Lambda\sin(\Psi_0 + \Phi_L)\end{aligned} \tag{5.28}$$

and Φ_L of (5.17) according to the estimates (5.27) with the definitions (5.26), (5.22).

From eqn.(5.25) one sees that (5.28) is only a reformulation of (5.19).

Using the estimates (5.20) and neglecting the contribution of the integral over $g(\theta)$ we can write uniform approximations $\tilde{v}^W_{c,s}(\theta)$ to two special solutions $\tilde{v}_{c,s}(\theta)$ of the variational equation (5.24) on the interval $(-\pi/2, -\varepsilon^{3/8-\delta})$:

$$\tilde{v}^W_{c/s}(\theta) = \frac{\varepsilon^{\kappa(t+\pi/2)}}{(-X_L(t))^{1/2}}\{\cos/\sin\}(\theta - \theta(-\pi/2)). \tag{5.29}$$

These solutions (and their approximants) obey: $\tilde{v}_c(\theta(-\pi/2)) = 1$, $\tilde{v}_s(\theta(-\pi/2)) = 0$, $d\tilde{v}_c/d\theta(\theta(-\pi/2)) = O(\varepsilon \ln(1/\varepsilon))$, $d\tilde{v}_s(\theta)/d\theta(\theta(-\pi/2)) = 1$. Comparing with (4.2) one recognizes in (5.29) the "‘WKB approximations"’ to solutions of eq.(5.24). At $\tau = -\varepsilon^{-\delta}$, $|X_L(t)| \approx \varepsilon^{1/8}\tau^{1/3} = \varepsilon^{1/8-\delta/3}$ and eq.(5.20) shows that:

$$|(\tilde{v}_{c,s} - \tilde{v}^W_{c,s})(\tau = \varepsilon^{-\delta})| = \varepsilon^{\kappa\pi/2+\delta/6-1/16} \times O\left(\int_{\theta(-\pi/2)}^{\varepsilon^{-4\delta/3}} g(\theta)d\theta\right) = O(\varepsilon^{\kappa\pi/2+r/2+\delta/6-1/16}) \qquad (5.30)$$

where we used $g(\theta) \approx 1/\theta^2$ (cf.(4.10)) and (5.22). The same quality of approximation holds for $d\tilde{v}_{c,s}/d\theta$, but for $d\tilde{v}_{c,s}/d\tau$ we obtain, in view of $d\tau/d\theta \approx \tau^{-1/3} = \varepsilon^{\delta/3}$:

$$|\frac{d\tilde{v}_{c,s}}{d\tau} - \frac{d\tilde{v}^W_{c,s}}{d\tau}| = O(\varepsilon^{\kappa\pi/2+r/2-\delta/6-1/16}) \qquad (5.31)$$

With the help of (5.29) a uniform approximation on $[-\pi/2, -\varepsilon^{-\delta}]$ of the solution of (5.24) with the initial conditions (5.28) reads:

$$\begin{aligned}\tilde{v}_L(t) \approx \tilde{v}^W_L(t) &\equiv \Lambda\varepsilon^{3/16+\kappa\pi/2}(\tilde{v}^W_c(\theta)\cos(\Psi_0 + \Phi_L) - \tilde{v}^W_s(\theta)\sin(\Psi_0 + \Phi_L)) \\ &= \frac{\Lambda\varepsilon^{3/16+\kappa\pi/2}}{(-X_L(t))^{1/2}}\varepsilon^{\kappa\pi/2+t}\cos(\theta(t) - \theta(-\pi/2) + \Psi_0 + \Phi_L)\end{aligned} \qquad (5.32)$$

With (5.31),(5.32) it follows that, at $\tau = -\varepsilon^{-\delta}$:

$$|(\tilde{v}_L - \tilde{v}^W_L)(\tau)| = O(\varepsilon^{1/8+\kappa\pi+r/2+\delta/6}) \qquad \left|\left(\frac{d\tilde{v}_L}{d\tau} - \frac{d\tilde{v}^W_L}{d\tau}\right)(\tau)\right| = O(\varepsilon^{1/8+\kappa\pi+r/2-\delta/6}) \qquad (5.33)$$

5.3. *The interval* $(-\varepsilon^{3/8-\delta}, 0)$

We notice that, while the "‘natural"’ order of magnitude for the boundary layer is $\varepsilon^{1/8}$(cf.eq.(1.18)), i.e. $u_L(\tau) = O(1)$, it turns out that, in fact, with the initial conditions (5.2), $u_L(\tau)$ is $O(\varepsilon^{\kappa\pi})$, i.e. for small enough ε - as argued in Sec.4.2 - it becomes smaller than the discontinuity of the reference solutions at $t = 0$. We compare now the evolution in the interval $(-\varepsilon^{3/8-\delta}, 0)$ of the solutions of the boundary layer equation with those of the variational equation around the *reference* solution $X_L(t)$, with the initial conditions (5.28). We introduce to this end "‘macroscopic"’ variables:

$$U(\tau) = \varepsilon^{-\kappa\pi}u_L(\tau), \qquad \tilde{U}(\tau) = \varepsilon^{-\kappa\pi}\tilde{u}_L(\tau) \qquad (5.34)$$

$U(\tau)$ satisfies :

$$\frac{d^2U}{d\tau^2} + 2\gamma\frac{dU}{d\tau} + 3\eta_L(\tau)^2U + 3\eta_L(\tau)\varepsilon^{\kappa\pi}U^2 + \varepsilon^{2\kappa\pi}U^3 = 0 \qquad (5.35)$$

and $\tilde{U}(\tau)$ is a solution of the linear part of (5.35). In (5.35), $\eta_L(\tau) = X_L(t)/\varepsilon^{1/8}$ and γ is given in (1.18). The solutions of (5.35) obeying initial conditions at $\tau = -\varepsilon^{-\delta}$ may be estimated by transforming (5.35) to an integral equation with the help of two independent solutions $U_1(\tau)$,$U_2(\tau)$ of the linear part. Using the method of "‘the variation of the parameters"’, we obtain:

$$\begin{aligned}U(\tau) &= A_1U_1(\tau) + A_2U_2(\tau) \\ &+ \int_{-\varepsilon^{-\delta}}^{\tau} \frac{(U_1(\tau)U_2(\tau\prime) - U_2(\tau)U_1(\tau\prime)}{W(U_1, U_2)}\varepsilon^{\kappa\pi}(U^2 + \varepsilon^{\kappa\pi}U^3)d\tau\prime\end{aligned} \qquad (5.36)$$

where A_1, A_2 are such that the values of U and $dU/d\tau$ at $\tau = -\varepsilon^{-\delta}$ are reproduced and $W(U_1, U_2)$ is the wronskian of U_1 and U_2. To discuss this equation, we need to know something about the solutions of:

$$\frac{d^2U}{d\tau^2} + 2\gamma\frac{dU}{d\tau} + 3\eta_L(\tau)^2U = 0 \qquad (5.37)$$

for τ in $(-\varepsilon^{-\delta}, 0)$. Writing:

$$U(\tau) = \exp(-\gamma\tau)V(\tau) \tag{5.38}$$

the equation for $V(\tau)$ is:

$$\frac{d^2V}{d\tau^2} + (3\eta_L(\tau)^2 - \gamma^2)V = 0 \tag{5.39}$$

In Appendix C we show we can choose two solutions of (5.39), called $V_{c,s}(\tau, \varepsilon)$, which for large $-\tau$ assume the "'WKB"' forms:

$$V_{c,s}^{(as)}(\tau, \varepsilon) = \frac{3^{1/4}}{\Xi(\tau,\varepsilon)^{1/4}} \{\cos / \sin\} \left(\int_{\tau}^{\tau_a} \Xi(\tau, \varepsilon)^{1/2} d\tau\right) \tag{5.40}$$

where

$$\Xi(\tau) = 3\eta_L(\tau)^2 - \gamma^2 \approx 3\eta_L(\tau, \varepsilon)^2 \approx 3\tau^{2/3} \tag{5.41}$$

for τ large and τ_a is an arbitrary (finite, ε-independent) value [16] of τ. These solutions may be extended down to $\tau = 0$ and have a well defined limit as $\varepsilon \to 0$. Indeed, in (5.39) and (5.41) $\eta_L(\tau, \varepsilon)$ is given by the inner expansion (3.9)(cf.eq.(3.34)). According to (3.9) and (3.11)(cf.eq.(3.34):

$$\eta_L(\tau, \varepsilon) = \eta_{00L}(\tau)(1 + O(\varepsilon^{3/4}\tau^2, \gamma/\tau^{5/3})) \tag{5.42}$$

so that the limit as $\varepsilon \to 0$ of expressions (5.40)at any finite τ is obtained by simply replacing $\eta_L(\tau, \varepsilon)$ by $\eta_{00L}(\tau)$. We can also pass to the limit in eq.(5.39) and one expects that its solutions tend to the solutions of its limiting form defined correspondingly by the requirement (5.40). Appendix C gives (straightforward) arguments for this. The solutions $V_{c,s}(\tau, \varepsilon, \tau_a)$ defined by (5.40) multiplied by $\exp(-\gamma\tau)$ (cf.(5.38)) are chosen as $U_{1,2}$ in (5.36). Clearly, the parameter τ_a is arbitrary and should drop out in the final expressions[17]. One verifies that, with this choice:

$$W(U_1, U_2) = \exp(-2\gamma\tau) \approx 1, \qquad \tau = O(e^{-\delta}) \tag{5.43}$$

Denoting then:

$$r(\tau) = U(\tau) - A_1U_1(\tau) - A_2U_2(\tau) \tag{5.44}$$

and using the fact that $|U_1(\tau)|, |U_2(\tau)| < M$ on $(-\varepsilon^{-\delta}, 0)$, for some M, we verify that the operator given by the integral on the right hand side of (5.36) defined on the space of functions $g(\tau)$ continuous on $[-\varepsilon^{-\delta}, 0]$ endowed with the $sup|(1+\tau^{1/6})g(\tau)|$ norm maps a ball of radius $const \times \varepsilon^{\kappa\pi-\delta/2}$ into itself and is, at least for small enough ε , a contraction, so that (5.36) admits of a unique solution there. Thus, the solution of the complete equation (5.35) departs from the solution of its linear part with the same initial conditions at $\tau = -\varepsilon^{-\delta}$ by quantities of $O(\varepsilon^{\kappa\pi-\delta/3})$ on $\left[-\varepsilon^{-\delta}, 0\right]$. For the derivatives one obtains estimates of $O(\varepsilon^{\kappa\pi-2\delta/3})$.

We still have to bound the evolution of the distance between two solutions of the linear equation (5.37) whose values differ by $O(\varepsilon^{r+\delta/6})$ and their derivatives by $O(\varepsilon^{r-\delta/6})$ at $\tau = -\varepsilon^{-\delta}$ (cf.eqns.(5.27) and (5.22)).This is a direct application of (C.19) in Appendix C, from which one deduces:

$$|\Delta U(\tau = 0)|, |\frac{d\Delta U}{d\tau}(\tau = 0)| = O(\varepsilon^r) \tag{5.45}$$

Recalling (5.34), we may summarize the foregoing by:

Lemma 5.3 *If a solution $v_L(t) = \varepsilon^{1/8}u_L(\tau)$ of the Duffing equation (4.1) differs from a solution $\tilde{v}_L(t) = \varepsilon^{1/8}\tilde{u}_L(t)$ of the variational equation around the reference solution $X_L(t)$ at $t = \varepsilon^{3/8-\delta}$ according to (5.27), then at $t = 0$:*

$$|(u_L - \tilde{u}_L)(\tau = 0)| = O(\varepsilon^{\kappa\pi+r-\delta/3}), \qquad |(\frac{du_L}{d\tau} - \frac{d\tilde{u}_L}{d\tau})(\tau = 0)| = O(\varepsilon^{\kappa\pi+r-2\delta/3}) \tag{5.46}$$

[16] It may be chosen as τ_0 of (4.2) but need not

[17] see Section 5.4

(recall $\delta = 3\kappa\pi/8$).

It is useful to recall the "'constitution"' of the exponents of ε in (5.46) which measure the order of magnitude of the approximation: a factor $\varepsilon^{\kappa\pi/2}$ originates in the initial conditions(5.28); a factor $\varepsilon^{\kappa\pi/2}$ is a result of the damping; these two factors control the magnitude of both $u_L(\tau)$ and $\tilde{u}_L(\tau)$ in (5.46); the remaining factor $\varepsilon^{r-\delta/3}$ (or $\varepsilon^{r-2\delta/3}$) states that the "'macroscopic"' quantities (5.34) are close to each other at $\tau = -\varepsilon^{-\delta}$ as stated in (5.27) and that this distance may increase a little as we move from $\tau = -\varepsilon^{-\delta}$ to $\tau = 0$.

5.4. *Summary of the quarter period map for* $\tau < 0$

The conclusion of section 5.3 is that to $O(\varepsilon^{\kappa\pi+r-2\delta/3})$ the Poincaré map $\mathbb{P}_{\mathbb{L}} : (v_L, dv_L/d\theta(pi/2)) \Rightarrow (u_L, du_L/d\tau)(\tau = 0)$ is given by the solutions of the variational equation around the reference solution $X_L(t)$ with initial conditions (5.28), where (Λ, Ψ_0) are related to $(v_L, dv_L/d\theta(t = -\pi/2))$ by (5.2) and Ξ_L is given by (5.41).We can write a more explicit form of $\mathbb{P}_{\mathbb{L}}$ using the combinations:

$$\begin{aligned}\tilde{V}_c &= \varepsilon^{\kappa\pi/2-1/16}(V_c\cos(\Omega(\tau_a)) - V_s\sin(\Omega(\tau_a)))\\ \tilde{V}_s &= \varepsilon^{\kappa\pi/2-1/16}(V_c\sin(\Omega(\tau_a)) + V_s\cos(\Omega(\tau_a)))\end{aligned} \tag{5.47}$$

with

$$\Omega(\tau_a) = \theta(\tau_a) - \theta(-\pi/2) \approx \int_{-\pi/(2\varepsilon^{3/8})}^{\tau_a} \Xi(\tau)^{1/2} d\tau \tag{5.48}$$

They are chosen so that $\varepsilon^{1/8}\tilde{V}_{c,s}$ are equal at large $\tau = -\varepsilon^{-\delta}$ to the WKB approximation (5.29) to the solutions $\tilde{v}_{c,s}$ of the variational equation (5.24), defined by initial conditions at $t = -\pi/2$ (see text following eq.(5.29)). The combinations (5.47) are, when multiplied by $\exp(-\gamma\tau)$, exact solutions of (5.37). The expression:

$$\begin{aligned}\tilde{v}_{aL}(t) \equiv& \Lambda\varepsilon^{3/16+\kappa\pi/2}(\tilde{V}_c\cos(\Psi_0 + \Phi_L) - \tilde{V}_s\sin(\Psi_0 + \Phi_L))\\ =& \Lambda\varepsilon^{\kappa\pi+1/8}(V_c\cos(\Omega(\tau_a) + \Psi_0 + \Phi_L) - V_s\sin(\Omega(\tau_a) + \Psi_0 + \Phi_L)) \equiv \varepsilon^{1/8}\tilde{u}_{aL}(t)\end{aligned} \tag{5.49}$$

is a solution of the variational equation (5.24) which differs at $\tau = -\varepsilon^{-\delta}$ from the solution $\tilde{v}_L(t)$ of Lemma 5.2 as described in (5.33). With the same argument used in eq.(5.45) and in Appendix C, this difference propagates down to $\tau = 0$: recalling the definitions of $\tilde{u}_L(t)$ (cf.Lemma 5.3 and (5.46)) and $\tilde{u}_{aL}(t)$ (cf.eq.(5.49)) then

$$|(\tilde{u}_L - \tilde{u}_{aL})(\tau = 0)|, \qquad \left|\left(\frac{d\tilde{u}_L}{d\tau} - \frac{d\tilde{u}_{aL}}{d\tau}\right)(\tau = 0)\right| = O(\varepsilon^{\kappa\pi+r/2}) \tag{5.50}$$

Thus, we can use the solution $\tilde{u}_{aL}(t)$, which involves the functions $V_{c,s}$ defined by means of (5.40) and having a well defined limit as $\varepsilon \to 0$ (cf.(5.42), to express the left hand side Poincaré mapping $\mathbb{P}_{\mathbb{L}}$ in a simpler form: taking (5.46) and (5.50) into account:

$$|u_L(\tau) - \tilde{u}_{aL}(\tau)| < |u_L(\tau) - \tilde{u}_L(\tau)| + |\tilde{u}_L(\tau) - \tilde{u}_{aL}(\tau)| = O(\varepsilon^{\kappa\pi+r-\delta/3}, \varepsilon^{\kappa\pi+r/2}) = O(\varepsilon^{\kappa\pi+r/2}) \tag{5.51}$$

and a similar estimate for the derivative; we can then state:

Theorem 5.1 *The quarter period Poincaré map* $\mathbb{P}_{\mathbb{L}} : (\Lambda, \Psi) \Rightarrow (u_L, du_L/d\tau(\tau = 0))$ *is given by:*

$$u_L(0) = \Lambda\varepsilon^{\kappa\pi}((V_c(0, \varepsilon, \tau_a)\cos(\Psi + \Phi_L + \Omega(\tau_a)) - V_s(0, \varepsilon, \tau_a)\sin(\Psi + \Phi_L + \Omega(\tau_a)) + O(\varepsilon^{r/2})) \tag{5.52a}$$

$$\begin{aligned}\frac{du_L}{d\tau}(0) = \Lambda\varepsilon^{\kappa\pi}(&\frac{dV_c}{d\tau}(0, \varepsilon, \tau_a)\cos(\Psi + \Phi_L + \Omega(\tau_a))\\ &- \frac{dV_s}{d\tau}(0, \varepsilon, \tau_a)\sin(\Psi + \Phi_L + \Omega(\tau_a)) + O(\varepsilon^{\kappa\pi/2}))\end{aligned} \tag{5.52b}$$

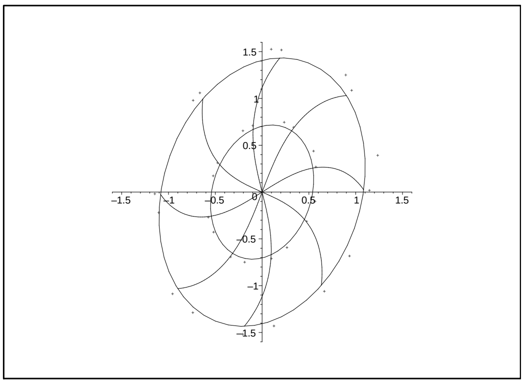

Figure 5: The image of the disk $\Lambda = 1$ under $\mathbb{P}_{\mathbb{L}}$ with the rays $\Psi = n\pi/4$

Recalling the definition (5.17) of Φ_L, eqns.(5.52a),(5.52b) show that circles of radius $\Lambda\varepsilon^{3/16+\kappa\pi/2}$ in the $(v_L(-\pi/2), dv_L/d\theta(-\pi/2))$ plane are mapped onto ellipses in the $(u_L(0),\ du_L/d\tau(0))$ plane, with a $\Lambda-$dependent phase Φ_L (see Fig.5). Thus the disk $\Lambda < const$ undergoes under $\mathbb{P}_{\mathbb{L}}$ a *torsion*. All quantities in (5.52a),(5.52b) may be obtained from a numerical solution of the variational equation only in an interval near $t = 0$: it is enough to find - for a given choice of τ_a - the solutions $V_{c,s}(\tau, \varepsilon, \tau_a)$ obeying the boundary condition given by (5.40) at some large $\tau = \varepsilon^{-\delta}$- and extend numerically the solution down to $\tau = 0$. One may object that the mapping seems to depend on an arbitrary parameter τ_a: this is, however, not the case. The reason is that in the form (5.32) of the solutions at large τ, the parameter τ_a does not appear at all. It is thus absent in the solutions $\tilde{V}_{c,s}$ of (5.47) which match (5.29) at $\tau = -\varepsilon^{-\delta}$. The rotations (5.47) which depend on τ_a leave the sums

$$A^2 \equiv V_c(0, \varepsilon, \tau_0)^2 + V_s(0, \varepsilon, \tau_a)^2, \qquad B^2 \equiv \left(\frac{dV_c}{d\tau}\right)^2 (0, \varepsilon, \tau_a) + \left(\frac{dV_s}{d\tau}\right)^2 (0, \varepsilon, \tau_a)$$

invariant. Further, the angle χ between the vectors (V_c, V_s) and $(dV_c/d\tau, dV_s/d\tau)$ is also invariant. The image of the circle $\Lambda = const$ under $\mathbb{P}_{\mathbb{L}}$ is

$$\frac{u_L^2}{A^2} + \frac{(du_L/d\tau)^2}{B^2} - 2\frac{u_L(du_L/d\tau)}{AB}\cos\chi = \Lambda^2 \sin^2\chi$$

It depends only on these three quantities, so that the independence of $\mathbb{P}_{\mathbb{L}}$ on τ_a is apparent. Moreover, the quantities $V_{c,s}(0, \varepsilon, \tau_a)$ and their derivatives have a limit as $\varepsilon \to 0$, according to the remarks surrounding (5.42) and to the discussion of Appendix C. In numerical calculations we choose τ_a = -10:at this value we can approximate : $\eta(\tau) \approx \sin(\tau\varepsilon^{3/8})^{1/3}/\varepsilon^{1/8}$. The limiting values for $\varepsilon = 0$ of the constants in (5.52a),(5.52b) are found to be:

$$V_c(0, 0, \tau_a = -10) \approx -1.163 \qquad \frac{dV_c}{d\tau}(0, 0, \tau_a = -10) \approx -0.178$$
$$V_s(0, 0, \tau_a = -10 \approx -0.0876 \qquad \frac{dV_s}{d\tau}(0, 0, \tau_a = -10) \approx -1.5086 \tag{5.53}$$

Then:

$$A = 1.1659, \qquad B = 1.5191, \qquad \chi = 78.960^\circ$$

6. The Right Hand Side Poincaré Map

6.1. *The continuation of the reference solution* X_L *to* $t > 0$

As one sees in Fig.4, the continuation of X_L to $t > 0$ traverses first the x-axis and then approaches the reference solution X_R as t increases, oscillating around it. We describe in the following this behaviour in more detail.

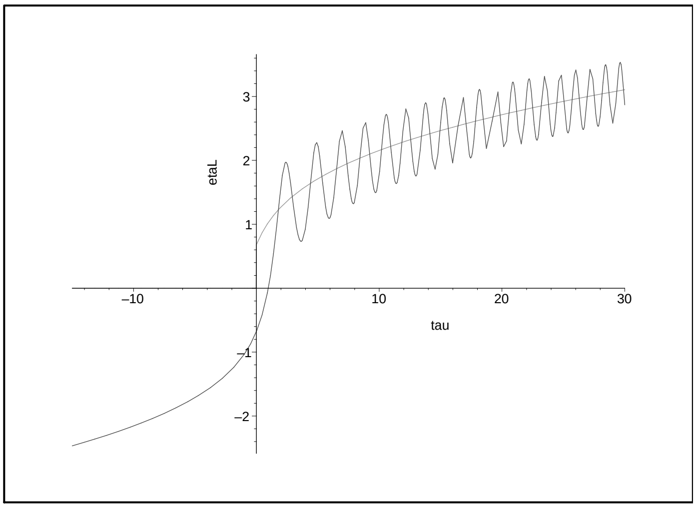

Figure 6: The continuation of the reference solution $\eta_L(\tau) = X_L(\tau)/\varepsilon^{1/8}$ to $t > 0$

The interval $0 < t < \tau_0\epsilon^{3/8}$

The initial values of the solution X_L at $\tau = 0$ are $(\varepsilon^{1/8}\eta_L(0), \varepsilon^{1/8}d\eta_L/d\tau(0)) \approx \varepsilon^{1/8}(-0.677, 0.472)$. In the limit $\epsilon \to 0$, the boundary layer equation

$$\frac{d^2\eta}{d\tau^2} + \eta^3 = \tau \tag{6.1}$$

admits of the symmetry $\tau \to -\tau, \eta \to -\eta$; as a consequence, it is true that $\eta_L(\tau) = -\eta_R(-\tau)$, where η_L, η_R are the solutions behaving like -$\tau^{1/3}, \tau^{1/3}$ as $\tau \to -\infty, \infty$ in turn. Therefore, in this limit, $\eta_L(0) = -\eta_R(0), d\eta_L/d\tau(0) = d\eta_R/d\tau(0)$ (cf.Section 3.3). For $t > 0$ (and any $\varepsilon > 0$), we denote $\Delta\eta(\tau, \varepsilon) = \eta_L - \eta_R$; it obeys the equation:

$$\frac{d^2\Delta\eta}{d\tau^2} + 2\gamma\frac{d\Delta\eta}{d\tau} + 3\eta_R^2\Delta\eta + 3\eta_R(\Delta\eta)^2 + (\Delta\eta)^3 = 0 \tag{6.2}$$

With the same argument as in Sect.4.1, eqn.(4.21) we verify that the energy associated with (6.2):

$$E(\tau) = \frac{1}{2}(\frac{d\Delta\eta}{d\tau})^2 + \frac{3}{2}\eta_R^2(\Delta\eta)^2 + \eta_R(\Delta\eta)^3 + \frac{(\Delta\eta)^4}{4} \tag{6.3}$$

obeys (if $E > 1$) the inequality:

$$\frac{dE}{d\tau} \leq const\frac{d\eta_R}{d\tau}E^{3/4} \tag{6.4}$$

from which one concludes that $E(\tau)$ and thus $\Delta\eta(\tau)$ are bounded at $t = \tau_0\epsilon^{3/8}$. Clearly, the same is true for all solutions starting in a disk of radius $\epsilon^{\kappa\pi}$ (or of any finite radius) around $(\Delta\eta(0), d\Delta\eta/d\tau(0))$. Here τ_0 is a time in the boundary layer region (typically $\tau_0 = 10$)

The interval $\tau_0\epsilon^{3/8} < t < \pi/2$

As in Sections 4.1 and 5.1 we write for $\Delta X(t)) \equiv \Delta\eta(t)\varepsilon^{1/8}$:

$$\Delta X(t) = \frac{\tilde{w}_{L,R}(\theta_R)}{X_R^{1/2}}\varepsilon^{\kappa t + 3/16} \tag{6.5}$$

with θ_R given in eq.(4.2). **Note:** *In the rest of this section we shall drop the subscript "'R"' on θ_R since it is clear that we confine ourselves to the time interval $[0,\pi/2]$. The subscript "'L"' is appended to quantities related to the continuation to $t>0$ of the left hand reference solution $X_L(t)$ or to departures from it*

The function $\tilde{w}_L$ obeys :

$$\frac{d^2\tilde{w}_{L,R}}{d\theta^2}+\tilde{w}_{L,R}(1+g(\theta))+k(\theta)\tilde{w}_{L,R}^2+\frac{1}{3}k(\theta)^2\tilde{w}_{L,R}^3=0 \tag{6.6}$$

where (cf.(4.14))

$$k(\theta)=\frac{\varepsilon^{3/16+\kappa t}}{X_R^{3/2}}=\frac{\exp(-\gamma\tau)}{\eta_R^{3/2}}, \tag{6.7}$$

$X_R(t)\equiv\varepsilon^{1/8}\eta_R(t(\theta))$ and $g(\theta)$ is given in (4.8) with the interchange $-X_L\leftrightarrow X_R$ As in Section 5.2, we move over to polar coordinates:

$$\tilde{w}_{L,R}=R_L(\theta)\cos(\theta+\phi_L(\theta)) \tag{6.8a}$$

$$\frac{d\tilde{w}_{L,R}}{d\theta}=-R_L\sin(\theta+\phi_L(\theta)) \tag{6.8b}$$

and obtain equations completely analogous to (5.5a),(5.5b) with the change $h\to -k$:

$$\frac{dR_L}{d\theta}=\frac{g(\theta)R_L(\theta)}{2}\sin(2z)+\frac{k(\theta)R_L^2(\theta)}{4}(\sin z+\sin 3z)+\frac{k(\theta)^2R_L(\theta)^3}{12}(\sin 2z+\frac{\sin 4z}{2}) \tag{6.9a}$$

$$\frac{d\phi_L}{d\theta}=\frac{g(\theta)}{2}(1+\cos 2z)+\frac{1}{4}k(\theta)R_L(\theta)(3\cos z+\cos 3z)+k(\theta)^2R_L(\theta)^2(\frac{3}{2}+2\cos 2z+\frac{1}{2}\cos 4z) \tag{6.9b}$$

with $z=\theta+\phi_L(\theta)$.We imitate now the arguments of Sect.5 concerning averaging and "'remove" first the terms in $k(\theta)$ in (6.9a), (6.9b) by a transformation of the dependent variables similar to eqs.(5.7a), (5.7b). One has to realize that, although the equations are similar to those of the previous section, the function $k(\theta)$ is of $O(1)$ when t is of $O(\varepsilon^{3/8})$, in opposition to $h(\theta)$ which is of $O(\varepsilon^{\kappa\pi})$ (cf.eq.(5.3)).However, it is monotonically decreasing with θ and arbitrarily small for large θ. According to Appendix B, the inversion of the transformations (5.7a), (5.7b) at fixed $\tau>\tau_0$ is possible if τ_0 is sufficiently large.

We obtain a set of equations for the functions $R_{1,L},\phi_{1,L}$ very similar to eqns.(5.8a), (5.8b). The initial conditions at τ_0 are different to O(1) from those for $R_L(\theta),\phi_L\theta$ in (6.9a), (6.9b). The boundedness of the solutions of these equations is not immediately apparent since $k(\theta)^2\approx(\theta_0/\theta)^{3/4}$ is not integrable (clearly, the integral over $k(\theta)^2$ is finite for finite ε but diverges as $\varepsilon\to 0$). We perform thus a second transformation, similar to (5.11a),(5.11b), which separates off a term $(-7/24)R_L(\theta)^2k(\theta)^2)$ in the equation for $\phi_{2,L}(\theta)$.

We denote:

$$\Phi_{L,R}(\theta)\equiv-\frac{7}{24}\int_{\theta_0}^{\theta}R_L(\theta)^2k(\theta)^2d\theta \tag{6.10}$$

Contrary to the phase Φ_L of eqn.(5.17), whose magnitude depends on the ratio $\varepsilon^{\kappa\pi}/\gamma^{1/3}$, the value at $\theta(\pi/2)$ of the additional phase $\Phi_{L,R}(\theta)$, eq.(6.10), appearing for t>0 in the oscillations of the extension of $X_L(t)$ around $X_R(t)$ is truly divergent as ε vanishes. Using the estimate :

$$\frac{2}{\pi}t\le\sin t\le t,\qquad\qquad 0<t<\pi/2$$

one verifies that:

$$\int_{\tau_0\varepsilon^{3/8}}^{\pi/2}k(\theta)^2\frac{d\theta}{dt}dt\approx\frac{1}{\varepsilon^{1/8}\kappa^{1/3}\ln^{1/3}(1/\varepsilon)}\int_0^{\infty}\frac{\exp(-2u)du}{u^{2/3}}=O(\gamma^{-1/3}) \tag{6.11}$$

As in Section 5.2 it turns out that $dR_L/d\theta$ is very small for large θ and thus we expect $R_L(\theta)$ to approach there a constant value $R_{L,f} \equiv R_L(t(\theta) = \pi/2)$. If this value is nonzero, the additional phase $\Phi_{L,R}$ is indeed divergent as $\varepsilon \to 0$. Now, the function $X_L(t)$ (and thus $\eta_L(\theta)$) still depends on the value of ε; so do the corresponding values $R_{L,f}$, for which we write for clarity $R_{L,f}(\varepsilon)$. In Appendix D we show

Lemma 6.1 *As $\varepsilon \to 0$, the values $R_{L,f}(\varepsilon)$ tend to a limit $R_{L,f}(0)$. This limit is obtained by solving eqns.(5.13a), (5.13b) with $-h(\theta)$ replaced by $k(\theta)$, eq.(6.7) where $\eta_R(\theta,\varepsilon)$ is replaced by $\eta_{00R}(\tau(\theta))$ of (3.24a) and γ is set equal to zero.*

For $\gamma = 0$ we obtain in (6.7) $k(\theta) = 1/\eta_R(\theta)^{3/2} \approx 1/\tau(\theta)^{1/2}$. A numerical evaluation leads to $R_f(0) \approx 0.84$. If we accept this as a "‘proof"’ that $R_f(0) \neq 0$, we can state

Lemma 6.2 *The value of the secular term $\Phi_{L,R}(\theta)$ at $\theta(t = \pi/2)$ is $O(\gamma^{-1/3})$.*

We define the "‘rest phase"’ left after the removal of the secular term as:

$$\bar{\phi}_L(\theta) \equiv \phi_L(\theta) - \Phi_{L,R}(\theta) \tag{6.12}$$

This "‘rest phase"’ is also ε-dependent (so that we should write $\bar{\phi}_{L,\varepsilon}(\theta)$), however, in a "‘harmless"’ manner:let $\bar{\phi}_{L,\varepsilon}(t(\theta) = \pi/2) \equiv \bar{\phi}_{L,f}(\varepsilon)$. With the same argument leading to Lemma 6.1 we show in Appendix D

Lemma 6.3 *As $\varepsilon \to 0$, the values $\bar{\phi}_{L,f}(\varepsilon)$ tend to a limit $\bar{\phi}_{L,f}(0)$. This limit is obtained by solving eqns.(5.13a),(5.13b) with $-h(\theta)$ changed to $k(\theta)$ of (6.7) with the same replacements as in Lemma 6.1.*

To conclude,taking (6.5) into account, the continuation of $X_L(t)$ to $t > 0$ oscillates around the reference solution $X_R(t)$; the departure from X_R reaches the value $0.84\varepsilon^{\kappa\pi/2+3/16}$ at $t = \pi/2$; as follows from the definition (4.2) of the variable θ_R, the number of oscillations with frequency proportional to $X_R(t)$ increases indefinitely as $\varepsilon \to 0$ and there is an additional phase, which also increases indefinitely in this limit, as shown by equation(6.11).

6.2. *The variational equation around* $X_L(t)$ *for* $t > 0$

In Section 5 we have seen that all those solutions of Duffing's equation which start at $t = -\pi/2$ in a disk of radius $\Lambda\varepsilon^{\kappa\pi/2+3/16}$ around the left hand side reference solution $X_L(t)$ land in a disk of radius $\varepsilon^{\kappa\pi}$ around the values $(\eta_L(0), d\eta_L/d\tau(0))$ (cf.Section 5.4, eqn.(5.52a), (5.52b); their departure from $X_L(t) \equiv \varepsilon^{1/8}\eta_L(\tau)$ is denoted there by $\varepsilon^{1/8}u_L(\tau)$. For $t > 0$, we consider the departures from $X_R(t) \equiv \varepsilon^{1/8}\eta_R(\tau)$:

$$u_R(\tau) = u_L(\tau) + \eta_L(\tau) - \eta_R(\tau) \equiv u_L(\tau) + \Delta\eta(\tau) \tag{6.13}$$

with $\Delta\eta(\tau)$ of (6.2) In analogy to eqn.(6.5), we write :

$$u_R(\theta) = \frac{w(\theta)\varepsilon^{\kappa t}}{\eta_R^{1/2}} \tag{6.14}$$

and define $R(\theta), \phi(\theta)$ in analogy to eqns.(6.8a), (6.8b):

$$w(\theta) = R\cos(\theta + \phi) \tag{6.15a}$$

$$\frac{dw}{d\theta} = -R\sin(\theta + \phi) \tag{6.15b}$$

The right hand side Poincaré map around $X_R(t)$ may then be written

$$\mathbb{P}_{\mathbb{R}} : \left(u_R(0), \frac{du_R}{d\tau}(0)\right) \Rightarrow (R(t(\theta) = \pi/2), \phi(t(\theta) = \pi/2)) \tag{6.16}$$

Since we are interested only in a small neighbourhood of the point $(\Delta\eta(0),\, d\Delta\eta/d\tau(0)) \approx (2\eta_L(0), 0)$, we expand the Poincaré map in a Taylor series around it:$(\phi_{L,f}(\varepsilon) \equiv \phi_L(t(\theta) = \pi/2, \varepsilon), cf.eq.(6.8b))$

$$\begin{aligned}\mathbb{P}_{\mathbb{R}}(u_R(0), du_R/d\tau(0)) = (R_{L,f}(\varepsilon), \phi_{L,f}(\varepsilon)) + \\ \mathbb{DP}_{\mathbb{R}}(\Delta\eta(0), d\Delta\eta/d\tau(0))(u_L(0), du_L/d\tau(0)) + ..\end{aligned} \tag{6.17}$$

where $(u_L(0), du_L/d\tau(0))$ is, according to eqn.(5.52a),(5.52b), $O(\varepsilon^{\kappa\pi})$. We write the mapping $\mathbb{P}_{\mathbb{R}}$ as a composition of three transformations:

$$\mathbb{P}_{\mathbb{R}} = \mathbb{P}_{\mathbf{f}} \circ \mathbf{T} \circ \mathbb{P}_{\mathbf{i}} \tag{6.18}$$

given by:

$$\mathbb{P}_{\mathbf{i}} \quad : \quad (u_R(0), du_R/d\tau(0)) \quad \Rightarrow \quad (u_R(\tau_0), du_R/d\tau(\tau_0)) \tag{6.19a}$$

$$\mathbf{T} \quad : \quad (u_R(\tau 0), du_R/d\tau(\tau_0)) \quad \Rightarrow \quad (R(\theta(\tau_0)), \phi(\theta(\tau_0))) \tag{6.19b}$$

$$\mathbb{P}_{\mathbf{f}} \quad : \quad (R(\theta(\tau_0)), \phi(\theta(\tau_0))) \quad \Rightarrow (R(\theta(t = \pi/2)), \phi(\theta(t = \pi/2))) \tag{6.19c}$$

where τ_0 is a "'time"' in the boundary layer already introduced in Section 6.1 following eq.(6.4); it is for convenience also chosen as origin of the variable θ_R, eq.(4.2). Corresponding to the composition (6.18) we write for the derivative:

$$\begin{aligned}\mathbb{DP}_{\mathbb{R}}(\Delta\eta(0), d\Delta\eta/d\tau(0)) = \mathbb{DP}_{\mathbf{f}}(R_L(\tau_0), \phi_L(\tau_0)) \circ \mathbb{D}\mathbf{T}(\Delta\eta(\tau_0), d\Delta\eta/d\tau(\tau_0)) \\ \circ\, \mathbb{DP}_{\mathbf{i}}(\Delta\eta(0), d\Delta\eta/d\tau(0))\end{aligned} \tag{6.20}$$

The elements of the jacobian matrices appearing in (6.20) are the values of solutions of the variational equation around $X_L(t)$ for $t > 0$ at τ_0 and $t = \pi/2$ with appropriate initial conditions. We evaluate next these elements.

The interval $0 < t < \tau_0 \varepsilon^{3/8}$

The variational equation around $X_L(t)$ reads:

$$\frac{d^2\delta u}{d\tau^2} + 2\gamma \frac{d\delta u}{d\tau} + 3\eta_L(\tau)^2 \delta u = 0 \tag{6.21}$$

On the bounded interval $0 \le \tau \le \tau_0$ this equation has bounded solutions; moreover, as $\varepsilon \to 0$, these solutions tend uniformly to those of the equation obtained by letting formally $\varepsilon = 0$ in (6.21).This means setting $\gamma = 0$ in (6.21) and replacing $\eta_L(\tau)$ by the continuation to $t > 0$ of the first term $\eta_{00L}(\tau)$ in the expansion (3.34).

The rapid oscillations of $\eta_L(\tau)$ for $\tau > 0$ (see Fig.6) lead to solutions of eq.(6.21) with a more complicated behaviour than those on the l.h.s.[18]. Whereas the image at $\tau = 0$ of the circle $\Lambda = 1$ at $t = -\pi/2$ is a (torsioned) ellipse (see Fig.5), the deformation of the latter under the flow for $\tau > 0$ is considerable ,see Fig.7, which shows the image of the (approximate) ellipse at $\tau = 0$ (crosses) at times $\tau = 3$ (boxes) and $\tau = 10$ (diamond) for values of $\varepsilon^{3/8} = 0.003$, $\kappa = 0.04$. The origin is now chosen at $(\eta_R(0), d\eta_R/d\tau(0))$.

[18]the corrections to the WKB formulae are determined by the function $g(\theta)$ (analog of eq.(4.10)) which contains the first and second derivatives of the rapidly oscillating $\eta_L(\tau)$

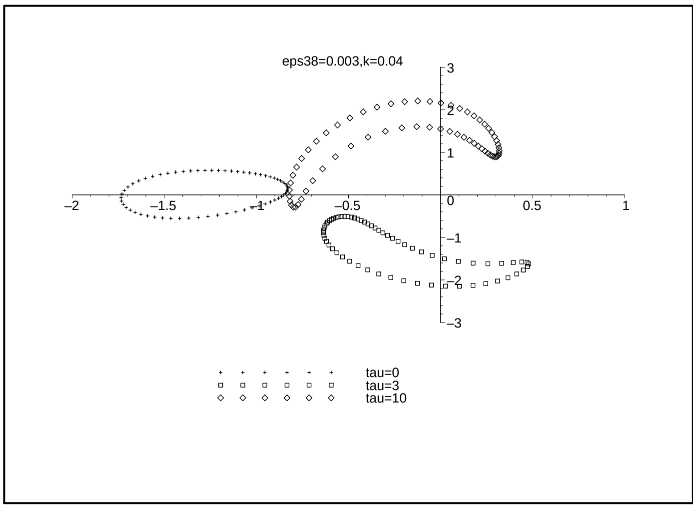

Figure 7: The deformation of the domain in Fig.5 at small $\tau > 0$

The transformation T

With the help of eqn.(6.14) and the definition of the variable θ_R, one verifies that the jacobian of the transformation:

$$\mathbf{T1} : (u_R, \frac{du_R}{d\tau})(\tau_0) \Rightarrow (w, \frac{dw}{d\theta})(\tau_0)$$

is:

$$\det \mathbb{D}\mathbf{T_1} = \frac{1}{\sqrt{3}} \exp(-\gamma\tau_0)$$

All elements of $\mathbb{D}\mathbf{T_1}$ are continuous at $\varepsilon = 0$. The supplementary transformation[19] $\mathbf{T2} : (w, dw/d\theta) \Rightarrow (R, \phi)$ given by eqns.(6.15a),(6.15b) has a jacobian equal to $-1/R(\tau_0)$; we take it as numerically established that $R(\tau_0) \neq 0$; it tends to a nonzero value $R_0(\tau_0)$ as $\varepsilon \to 0$.

The interval $\tau_0\varepsilon^{3/8} < t < \pi/2$

The jacobian matrix $\mathbb{D}\mathbb{P}_\mathbf{f}$ is more complicated: it contains elements which diverge as $\varepsilon \to 0$: the reason is that the contribution coming from the variation of the term $\Phi_{L,R}$ of (6.10) (see Lemma 6.2) is divergent in this limit. It is thus convenient to study first a "‘reduced"’ transformation:

$$\mathbb{P}_\mathbf{fr} : (R, \phi)(\tau_0) \Rightarrow (R, \bar{\phi})(t = \pi/2) \tag{6.22}$$

where $\bar{\phi}$ is the "‘rest phase"’ defined for each solution in analogy to (6.12),by subtraction of the "‘secular"’ term. To this transformation we associate the jacobian matrix $\mathbb{D}\mathbb{P}_\mathbf{fr}$. Concerning it, we show:

Lemma 6.4 *The matrix elements of* $\mathbb{D}\mathbb{P}_{\mathbf{fr}}$ *are bounded and continuous with respect to* ε *as* $\varepsilon \to 0$.

Proof. The proof is similar to the one in Appendix D and is based on a qualitative study of the solutions of the variational equation around the solutions $R_{4L,0}, \phi_{4L,0}$ of eqns.(D.4a), (D.4b) and of the solutions $R_{4L,\varepsilon}$, $\phi_{4L,\varepsilon}$ of (D.6a), (D.6b), in turn . According to the choice of initial conditions, these solutions, denoted in the following generally by $\delta R(\theta), \delta\phi(\theta)$,may be identified with the partial derivatives $\partial R(\theta)/\partial R_0$, $\partial\phi(\theta)/\partial R_0$ (if $\delta R(0) = 1, \delta\phi(0) = 0$) or $\partial R(\theta)/\partial\phi_0$, $\partial\phi(\theta)/\partial\phi_0$ (if $\delta R(0) = 0, \delta\phi(0) = 1$), where R_0, ϕ_0 are the initial values at $\theta(\tau_0) = 0$.

[19] $\mathbf{T} = \mathbf{T}_2 \circ \mathbf{T}_1$

The proof consists of three steps: (i)we show that, if ε is set formally to zero, then the solutions $(\delta R^0)(\theta), (\delta\bar{\phi}^0)(\theta)$ (in the latter the "‘secular term "‘ has been removed, see (6.33) below) of the variational equation around $X_{L0} \equiv \eta_{00L}(\tau)\varepsilon^{1/8}$ approach a limit as τ increases indefinitely; (ii)For finite ε we show that, if $\tau > \varepsilon^{-\alpha}$, for some $\alpha < 3/8$, the solutions $(\delta R)(\theta), (\delta\bar{\phi})(\theta)$ differ by arbitrarily small amounts from their values at $t = \pi/2$, provided ε is appropriately small;(iii) We show that, for an interval of values of $\tau, \tau_0 < \tau < \varepsilon^{-\beta}$, with $3/8 > \beta > \alpha$, the difference between the solutions $\delta R^0(\theta), \delta\bar{\phi}^0(\theta)$ (corresponding to $\varepsilon = 0$) and $\delta R(\theta), \delta\bar{\phi}(\theta)$ (corresponding to a finite ε) may be made as small as we wish, by letting ε be appropriately small.

In proving (i), we append for clarity to all variables related to the variational equation a superscript "‘0"’,recalling that ε is set equal to zero. We assume we have "‘removed"’ by successive transformations of the dependent variables the terms of $O((k^0)^3)$ and $O(dk^0/d\theta)$,$(k^0(\theta) \equiv 1/\eta_{00}(\theta)^{3/2})$ and consider first the variational equation around the solutions $R_{4L,0}, \phi_{4L,0}$(eqns.(D.4a),(D.4b)):

$$\frac{d(\delta R_4^0)}{d\theta} = k^0(\theta)^4 R_{L,0}(\theta)^4 \bar{P}_1(z_0)(\delta R^0)(\theta) + k^0(\theta)^4 R_{L,0}(\theta)^5 \bar{P}_2(z_0)(\delta\phi^0)(\theta) + ... \tag{6.23a}$$

$$\begin{aligned}\frac{d(\delta\phi_4^0)}{d\theta} &= -\frac{7}{24}\delta(k^0(\theta)^2 R_{L,0}(\theta)^2) + k^0(\theta)^4 R_{L,0}(\theta)^3 \bar{Q}_1(z_0)(\delta R^0)(\theta) \\ &+ k^0(\theta)^4 R_{L,0}(\theta)^4 \bar{Q}_2(z_0)(\delta\phi^0)(\theta) + ...\end{aligned} \tag{6.23b}$$

where $\bar{P}_{1,2}$, $\bar{Q}_{1,2}$ are trigonometric polynomials of $z_0 = \theta + \phi_{L,0}$ and the dots stay for terms which fall off more rapidly with θ. It is relevant that the polynomials $\bar{P}_1(z_0)$, $\bar{P}_2(z_0)$, $\bar{Q}_2(z_0)$ have zero mean whereas $\bar{Q}_1(z_0)$ contains a constant ("‘secular"’) term. From the form of the transformations leading from $R_{L,0}, \phi_{L,0}$ to $R_{4L,0}, \phi_{4L,0}$ (cf.eqs(5.7a),(5.7b), (5.9a), (5.9b), (5.11a),(5.11b)) we see that:

$$\delta R^0(\theta) = \delta R_4^0(\theta)(1 + O(k^0(\theta))) + \delta\phi_4^0(\theta)O(k^0(\theta)) \tag{6.24a}$$

$$\delta\phi^0(\theta) = \delta R_4^0(\theta)O(k^0(\theta)) + \delta\phi_4^0(\theta)(1 + O(k^0(\theta)) \tag{6.24b}$$

In (6.24a), (6.24b) the terms denoted by $O(k^0)$ may be read off (5.9a),(5.9b) : they contain trigonometric polynomials with zero mean. It follows that the variation of the secular term in (6.23b) is

$$\delta(k^0(\theta)^2 R_{L,0}(\theta)^2) = 2k^0(\theta)^2 R_{L,0}(\theta)\delta R_4^0(\theta) + (O(k^0(\theta)^3)\delta\phi_4^0(\theta) \tag{6.25}$$

These terms also give the leading order (the terms with the slowest falloff in θ) of the coefficients of $\delta R_4^0(\theta)$, $\delta\phi_4^0(\theta)$ in (6.23b).

We show next that the solution $\delta R_4^0(\theta)$ of eqn.(6.23a) is actually bounded as $\theta \to \infty$ and the solution $\delta\phi_4^0$ of eqn.(6.23b) obeys $\delta\phi_4^0 = O(\theta^{1/4})$ in the same limit. We consider to this end the Lyapunov expression:

$$L = r(\delta R_4^0(\theta))^2 + s(\delta\phi_4^0(\theta))^2/\theta^{1/2} \tag{6.26}$$

where $r, s > 0$ are two parameters which may be chosen freely. Denoting by $a_R(\theta), a_P(\theta)$ and $b_R(\theta), b_P(\theta)$ the coefficients of $\delta R_4^0, \delta\phi_4^0$ in eqns.(6.23a),(6.23b) in turn, we may write, after taking (6.25) into acount:

$$\begin{aligned}\frac{1}{2}\frac{dL}{d\theta} &= ra_R(\theta)(\delta R_4^0)^2 + (ra_P(\theta) + s\frac{b_R(\theta)}{\theta^{1/2}})(\delta R_4^0)(\delta\phi_4^0) \\ &+ s\frac{b_P(\theta)}{\theta^{1/2}}(\delta\phi_4^0)^2 - \frac{1}{4}s\frac{(\delta\phi_4^0)^2}{\theta^{3/2}}\end{aligned} \tag{6.27}$$

Using now the inequalities:

$$|\delta R_4^0| \leq \sqrt{\frac{L}{r}} \qquad |\delta\phi_4^0| \leq \theta^{1/4}\sqrt{\frac{L}{s}} \tag{6.28}$$

and the fact that the last term in eqn(6.27) is negative, we deduce:

$$\frac{dL}{d\theta} \leq 2L(|a_R(\theta)| + \sqrt{\frac{r}{s}}\theta^{1/4}|a_P(\theta)| + \sqrt{\frac{s}{r}}\frac{|b_R(\theta)|}{\theta^{1/4}} + b_P(\theta)) \tag{6.29}$$

For large θ the dominant term is the one containing $|b_R(\theta)|/\theta^{1/4}$ as one sees by reverting to eqns.(6.23a), (6.23b) and (6.25): it falls off like $1/\theta$. Inequality (6.29) can be integrated to yield

$$L < const \times \theta^{2C\sqrt{s/r}}, \qquad C = sup|b_R(\theta)|\theta^{3/4}$$

We may choose r and s so that the exponent of θ be $2d$, with $0 < d < 1/8$. It follows then from eqn.(6.28) that:

$$|\delta\phi_4^0(\theta)| < const \times \theta^{d+1/4} \tag{6.30}$$

We return now to eqn(6.23a) and integrate it from an initial value θ_i to some θ: since $\delta\phi_4^0$ obeys (6.30), the integral over $\delta\phi_4^0$ is convergent and we deduce from Gronwall's inequality that:

$$\delta R_4^0(\theta) < const \times \exp(\int_{\theta_i}^{\theta} |a_R(\theta)|d\theta) < const \tag{6.31}$$

Thus $\delta R_4^0(\theta)$ is bounded for all θ. Further, integrating eqn.(6.23a) between two values θ_m and θ_n one sees that $|\delta R_4^0(\theta_m) - \delta R_4^0(\theta_n)| \to 0$ as $(\theta_m, \theta_n) \to \infty$ and thus $\delta R_4^0(\theta)$ approaches a limit as $\theta \to \infty$. This limit is approached like $1/\theta^{1/4-d}$ (The dominant term in (6.23a) is the one containing $\delta\phi^0$). From (6.24a) and the fact that $k^0(\theta) \approx 1/\theta^{3/8}$, it follows that, if the exponent d in (6.30) is less than 1/8, $\delta R^0(\theta)$ approaches itself a limit as $\theta \to \infty$. This approach is at least of $O(1/\theta^{1/8-d})$.

We apply next Gronwall's inequality to eq.(6.23b), after taking (6.25) into account:

$$|\delta\phi_4^0(\theta)| < \sup|\int_0^{\theta} k^0(\theta)^2 R_{L,0}(\theta)\delta R(\theta)d\theta| \exp(const \times \int_0^{\theta} k^0(\theta)^3 d\theta) < const \times \theta^{1/4} \tag{6.32}$$

where we have used the fact that the integral in the exponent is convergent, whereas the factor in front is bounded by $const \times \theta^{1/4}$ in view of eq.(6.31).As a consequence of (6.32) we may assume from now on that the small exponent $d = 0$. We define further:

$$\delta\bar{\phi}_4^0(\theta) = \delta\phi_4^0(\theta) + \frac{7}{12}\int_0^{\theta} k^0(\theta)^2 R_{L,0}(\theta)(\delta R^0)(\theta)d\theta \tag{6.33}$$

Taking into account eqns.(6.24a),(6.24b),it is true that:

$$\begin{gathered}\frac{d\delta\bar{\phi}_4^0(\theta)}{d\theta} = c_R(\theta)(\delta R_4^0)(\theta) + c_P(\theta)(\delta\phi_4^0)(\theta)) \\ c_R(\theta), c_P(\theta) = O(k^0(\theta)^4)\end{gathered} \tag{6.34}$$

We integrate eqn.(6.34) from an initial value θ_i to θ and use the bounds of eqn.(6.31), (6.32) to deduce that $\delta\bar{\phi}_4^0(\theta)$ is itself bounded and has a limit as $\theta \to \infty$. We can now revert to an "‘original"’ variation of the "‘rest phase"’ $\delta\bar{\phi}^0$ defined by :

$$\delta\bar{\phi}^0 = \delta\phi^0 + \frac{7}{12}\int_0^{\theta} k^0(\theta)^2 R_{L,0}(\theta)\delta R^0(\theta) \tag{6.35}$$

Since $\delta\phi^0$ is related to $\delta\phi_4^0$ by (6.24b) and $\delta\bar{\phi}_4^0$ defined in (6.33) was shown to be bounded it follows that $\delta\bar{\phi}^0$ is itself bounded and has a limit as $\theta \to \infty$, which is approached like $1/\theta^{1/8}$ (or $1/\tau^{1/6}$).We denote the limits of $\delta R^0(\theta), \delta\bar{\phi}^0(\theta)$ by $\delta R_f^0, \delta\bar{\phi}_f^0$.

We now turn to point(ii) of the proof and show for finite ε that, if $\tau > \varepsilon^{-\alpha}$ with $0 < \alpha < 3/8$, the difference between $\delta R(\theta(\tau);\varepsilon) \equiv \delta R(\theta)$ and its value at $\theta(t = \pi/2)$ becomes vanishingly small as $\varepsilon \to 0$. The same is true for $\delta\bar{\phi}(\theta;\varepsilon) \equiv \delta\bar{\phi}(\theta)$ of (6.35). This is done by repeating the argument above but using the complete function $k(\theta) \equiv k(\theta,\varepsilon)$ for finite ε and correspondingly a modified Lyapunov function $\mathcal{L}(\theta)$:

$$\mathcal{L}(\theta) = r(\delta R_4(\theta))^2 + s\frac{\delta\phi_4(\theta)^2}{\Phi(\theta)^2}, \qquad \Phi(\theta) \equiv \int^{\theta} k(\theta)^2 d\theta \tag{6.36}$$

The function $\Phi(\theta)$ increases like $\theta^{1/4}$ for $\tau < 1/\gamma$ and then stays approximately constant at values of $O(1/\gamma^{1/3})$. All arguments used above for $\varepsilon = 0$ may be repeated with the conclusion that

$$|\delta R_4(\theta)| < const \qquad\qquad |\delta\phi_4| < const \times \Phi(\theta) < const \times \theta^{1/4} \tag{6.37}$$

and that the differences to the values at $t = \pi/2$ obey:

$$|\delta R_4(\tau = \varepsilon^{-\alpha}) - \delta R_4(t = \pi/2)|, |\delta\bar{\phi}_4(\tau = \varepsilon^{-\alpha}) - \delta\bar{\phi}_4(t = \pi/2)| \leq C\varepsilon^{\alpha/3} \tag{6.38}$$

Returning to the original $\delta R(\theta), \delta\bar{\phi}$, these bounds are turned into:

$$|\delta R(\tau = \varepsilon^{-\alpha}) - \delta R(t = \pi/2)|, |\delta\bar{\phi}(\tau = \varepsilon^{-\alpha}) - \delta\bar{\phi}(t = \pi/2)| \leq C\varepsilon^{\alpha/6} \tag{6.39}$$

We turn now to point (iii) of the argument and compare directly the values of $\delta R(\theta,\varepsilon)$, $\delta\bar{\phi}(\theta,\varepsilon)$ with those obtained for $\varepsilon = 0$. To this end, we revert again to the equations (6.23a),(6.23b) and (6.34) written appropriately for $\varepsilon = 0$ and a finite small ε value. We subtract them and using notations like, e.g.: $\Delta(\delta R_4(\theta)) \equiv \delta R_4(\theta,\varepsilon) - \delta R_4^0(\theta)$ and similarly for $\Delta(\delta\bar{\phi}_4), \Delta(\delta\phi_4)$ and also $\Delta a_R(\theta) \equiv a_R(\theta,\varepsilon) - a_R(\theta,0)$, etc.(cf.(6.29)) we obtain equations of the form:

$$\frac{d\Delta(\delta R_4)}{d\theta} = \Delta a_R(\theta)\delta R_4(\theta) + \Delta a_P(\theta)\delta\phi_4(\theta) + a_R^0(\theta)\Delta\delta R_4(\theta) + a_P^0(\theta)\Delta\delta\phi_4(\theta) \tag{6.40a}$$

$$\frac{d\Delta(\delta\phi_4)}{d\theta} = \Delta b_R(\theta)\delta R_4(\theta) + \Delta b_P(\theta)\delta\phi_4(\theta) + b_R^0(\theta)\Delta\delta R_4(\theta) + b_P^0(\theta)\Delta\delta\phi_4(\theta) \tag{6.40b}$$

These equations are an inhomogeneous version of eqns.(6.23a), (6.23b). The solutions of the homogeneous part have been shown to obey the bounds of eqns.(6.30),(6.31). By means of the method of "‘variation of the constants"’ we may write qualitatively a general solution of eqns.(6.40a),(6.40b): since the initial conditions are $\Delta\delta R_4(0) = \Delta\delta\phi_4(0) = 0$ (we are interested in those solutions that obey,e.g.$\partial R/\partial R_0(\theta = 0) = 1$,independently of ε), we obtain, e.g.

$$\begin{aligned}\Delta\delta R(\theta) = &\int_0^{\theta} (f_R(\theta\prime)(\delta\phi^{(2)})(\theta\prime) - f_\phi(\theta\prime)(\delta R^{(2)})(\theta\prime))d\theta\prime(\delta R^{(1)})(\theta) \\ &- \int_0^{\theta} (f_R(\theta\prime)(\delta\phi^{(1)})(\theta\prime) - f_\phi(\theta\prime)(\delta R^{(1)})(\theta\prime))d\theta\prime(\delta R^{(2)})(\theta)\end{aligned} \tag{6.41}$$

where $f_R(\theta), f_\phi(\theta)$ represent the inhomogeneous terms in eqns.(6.40a), (6.40b) and $\delta R^{(i)}$, $\delta\phi^{(i)}$, $i = 1, 2$ are two independent solutions of the homogeneous equation with a wronskian equal to unity. As an example of an estimate of the differences Δa_R,Δb_R, etc. we consider:

$$\Delta b_R(\theta) = O(\frac{1}{\tau} - \frac{\exp(-\gamma\tau)}{\tau(1 + \varepsilon^{3/4}\tau^2)}) = O(\varepsilon^{3/4}\tau), \qquad \tau < 1/\gamma \tag{6.42}$$

These differences are to be evaluated at fixed θ, i.e. the values of τ appearing in the two terms in (6.42) are *a priori* different; however, as shown in Appendix D (cf.eq.(D.12) it is enough for coarse estimates to use in both terms the value of τ corresponding to $\varepsilon = 0$, if $\tau < \varepsilon^{-\delta}$ with $\delta < 3/8$. Using eqns.(6.30),(6.31) one verifies that :

$$f_R(\theta) = O(\varepsilon^{3/4}\tau^{1/3}), \qquad f_\phi(\theta) = O(\varepsilon^{3/4}\tau) \tag{6.43}$$

With this we estimate from eqn.(6.41) and its analogue for $\Delta\delta\phi_4$:

$$\Delta\delta R_4(\theta) = O(\varepsilon^{3/4}\theta^{7/4}), \qquad \Delta\delta\phi_4(\theta) = O(\varepsilon^{3/4}\theta^2) \tag{6.44}$$

Taking into account eqns.(6.24a),(6.24b), we verify that these estimates hold unchanged even for $\Delta\delta R(\theta), \Delta\delta\phi(\theta)$. Using eqn.(6.35), we obtain:

$$\Delta\delta\bar{\phi} = O(\varepsilon^{3/4}\theta^{7/3})$$

If $\tau < \varepsilon^{-\beta}$,$(\theta < \varepsilon^{-4\beta/3})$, $\Delta\delta R$, $\Delta\delta\bar{\phi}$ tend to zero with ε provided $3/4 - 28\beta/9 < 0$; this is fulfilled if,e.g. $\beta = 1/5$. For any choice of $\alpha < 1/5$, the difference between $\delta R, \delta\bar{\phi}(\tau)$ and their values at $t = \pi/2$ vanishes as $\varepsilon \to 0$ (cf.eqns.(6.38),(6.39)). Acording to (i) $\delta R^0(\theta), \delta\bar{\phi}^0(\theta)$ approach their asymptotic values like $1/\theta^{1/8}$ and thus the differences to these latter are $O(\varepsilon^{\alpha/6})$ at $\tau = \varepsilon^{-\alpha}$. We can thus conclude the limiting values $(\delta R(t = \pi/2), \delta R^0), (\delta\bar{\phi}(t = \pi/2), \delta\bar{\phi}^0)$ also approach each other as $\varepsilon \to 0$. This ends the proof of Lemma 6.4.

6.3. *The mapping* $\mathbb{DP}_\mathbb{R}$

The first derivative $\mathbb{DP}_\mathbb{R}$ of the Poincaré map is obtained from the values at $\tau = \pi/(2\varepsilon^{3/8})$ of two special solutions $\delta R(\theta), \delta\phi(\theta)$ of the variational equation, with initial conditions at $\tau = \tau_0$:$(\delta R = 1, \delta\phi = 0), (\delta R = 0, \delta\phi = 1)$ in turn. According to Lemma 6.4 the values $\partial R/\partial R_0(\pi/(2\varepsilon^{3/8}))$, $\partial R/\partial\phi_0(\pi/(2\varepsilon^{3/8}))$ approach as $\varepsilon \to 0$ the asymptotic values $(\partial R/\partial R_0)^0$, $(\partial R/\partial\phi_0)^0$ of the solutions of an equation in which ε was set formally equal to zero (we recall $R_0 \equiv R(\tau_0), \phi_0 \equiv \phi(\tau_0)$, with R, ϕ of eq.(6.15a),(6.15b)). These values turn out to be:

$$A_R^0 \equiv \left(\frac{\partial R}{\partial R_0}\right)^0 \cong 0.96 \qquad A_P^0 \equiv \left(\frac{\partial R}{\partial \phi_0}\right)^0 \cong 0.08 \tag{6.45}$$

The same is true for the asymptotic values of the derivatives of the "‘rest phase"’ $\bar{\phi}^0$

$$B_R^0 \equiv \left(\frac{\partial \bar{\phi}}{\partial R_0}\right)^0 \cong 0.25 \qquad B_P^0 \equiv \left(\frac{\partial \bar{\phi}}{\partial \phi_0}\right)^0 \cong 1.08 \tag{6.46}$$

These values depend on the point τ_0 which, for numerical convenience, is chosen sufficiently large so that the asymptotic form $\eta(\tau) \approx \tau^{1/3}$ be valid [20]. It turns out that the convergence of the variations $\delta R(\theta)$ to the limiting values is rapid, that of the $\delta\bar{\phi}$ is, however, very slow.

On the other hand, the derivative $\partial\phi/\partial R_0(\theta)$ diverges as $\theta \to \infty$. This shows the origin of the asymptotic circle map given by the Duffing equation: the small disk of radius $\varepsilon^{\kappa\pi}$ is "‘stretched"’ in a τ-time interval of the order $1/\gamma$ in a rectangle in the (R, ϕ) plane, highly elongated in the ϕ-direction. In a τ - interval of $O(1/\gamma)$, friction plays no role: this stretching is entirely "‘hamiltonian"’: the volume in phase space is conserved (the transformation (6.15a),(6.15b) is not canonical;for a unit jacobian, one must multiply $\delta\phi$ by R; the area of the rectangle is multiplied by a factor $R(\tau = 1/\gamma)/R(\tau_0) \approx 1$). Clearly, when ϕ is "‘wrapped"’ back on the unit circle, it will possibly cover it - depending on the size of the initial disk - more than once: it is this mapping of the circle into itself which leads to the chaotic motion observed at smaller values of the damping[21].The arguments of this paper show that it must be observed at increasing values of the damping as the forcing Γ increases indefinitely.

[20] $\tau_0 = 10$ is a possible choice; clearly, its value drops out in the final results

[21] As is shown in Section 7, chaotic motions appear long before the circle is completely covered

In Fig.8 we show the image of the circle $\Lambda = 1$ at $t = -\pi/2$ in the $(w, dw/d\theta)$ plane at $t = \pi/2$ for $\varepsilon^{3/8} = 0.003, \kappa = 0.04$; one sees the extreme angular stretching (in the ϕ-direction) caused by the diverging derivative $\partial\phi/\partial R_0$. The crosses show the approximation offered by the variational equation (for $\partial R, \phi/\partial R_0, \phi_0$). Fig.9 shows a situation at smaller damping ($\varepsilon^{3/8} = 0.002, \kappa = 0.02$) where the

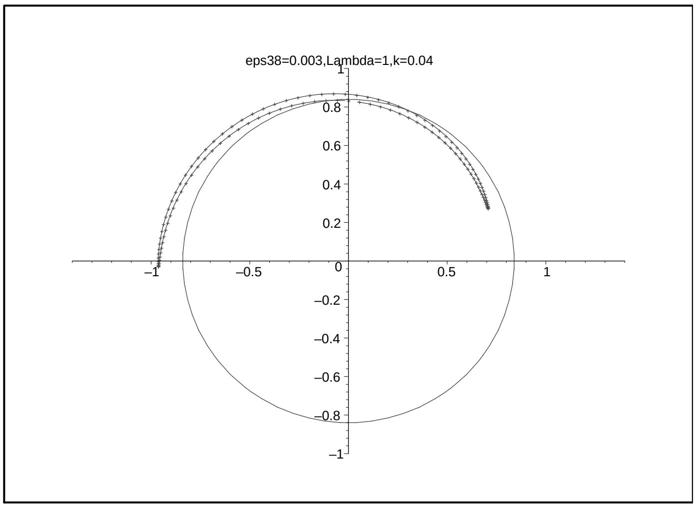

Figure 8: The image of the circle $\Lambda = 1$ at $t = -\pi/2$ in the $(w, dw/d\theta)$ plane at $t = \pi/2$

stretching exceeds 2π. At smaller ε (larger forcing) the arms of the spiral approach each other so that asymptotically the disk $\Lambda = 1$ is mapped into a very thin ring at t=$\pi/2$. The points show again the approximation of the Poincaré plot by the first derivative; feeling supported by this numerical evidence[22], we do not discuss in this paper at all the corrections due to higher terms of the Taylor expansion.

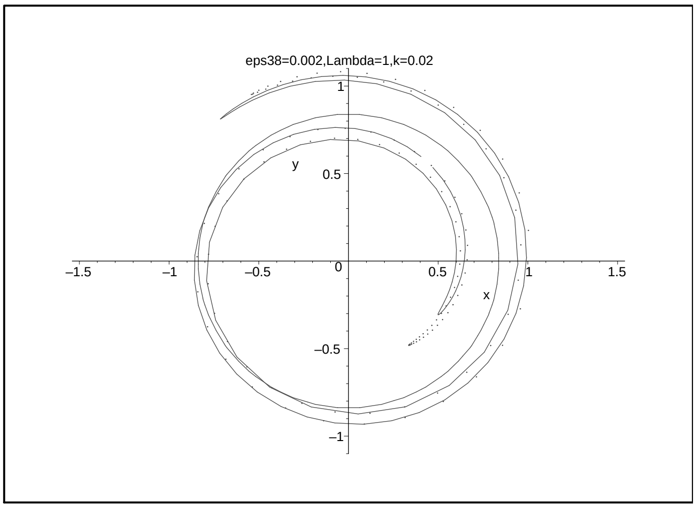

Figure 9: The image of the circle $\Lambda = 1$ at $t = -\pi/2$ in the $(w, dw/d\theta)$ plane at $t = \pi/2$ at smaller damping

We evaluate next more carefully the derivatives $\partial\phi/\partial R_0(\theta(\pi/2))$, $(\partial\phi/\partial\phi_0)(\theta(\pi/2))$ for finite, small values of ε.

$$\frac{\partial\phi}{\partial R_0} = \frac{\partial\bar{\phi}}{\partial R_0} - \frac{7}{12}\int_0^{\theta(\pi/2)} k(\theta)^2 R_{L,\varepsilon}(\theta)\frac{\partial R}{\partial R_0}(\theta)d\theta \tag{6.47}$$

[22]The agreement is better than warranted by the estimates of the paper

Both the first term and $R_{L,\varepsilon}(\theta), (\delta R)(\theta)$ under the integral sign are bounded [23] at $\theta = \theta(\pi/2)$. The leading behaviour of the phase is determined by the integral: according to Lemma 6.1 and Lemma 6.4, the values of $R_{L,\varepsilon}(\theta(\pi/2))$,$\partial R/\partial R_0(\theta(\pi/2)$ have limits as $\varepsilon \to 0$, denoted by $R_{L,f}(0)$, A_R^0 (vgl.Lemma 6.1 and 6.45). Thus we may write for the dominant contribution in (6.47):

$$\left|\left(\frac{\partial\phi}{\partial R_0}\right)_d\right| = \left|-\frac{7}{12}R_{L,f}(0)A_R^0\int_0^{\theta(\pi/2)}k(\theta)^2 d\theta\right| \leq const \times \int_{\tau_0}^{\pi/(2\varepsilon^{3/8})}\frac{\exp(-2\gamma\tau)}{\tau^{2/3}}d\tau$$
$$= \frac{const}{\gamma^{1/3}} \tag{6.48}$$

which shows explicitly the divergence as $\varepsilon \to 0$ in the factor $1/\gamma^{1/3}$. It turns out that the factor multiplying $1/\gamma^{1/3}$ has itself a slow logarithmic dependence on ε and approaches a finite value as $\varepsilon \to 0$ (≈ 1.80). Fig.10 shows the ε-dependence of the quantity $C_0(\varepsilon)$ defined by comparison to (6.48) through

$$\left(\frac{\partial\phi}{\partial R_0}\right)_d = -\frac{C_0(\varepsilon)A_R^0}{\gamma^{1/3}} \tag{6.49}$$

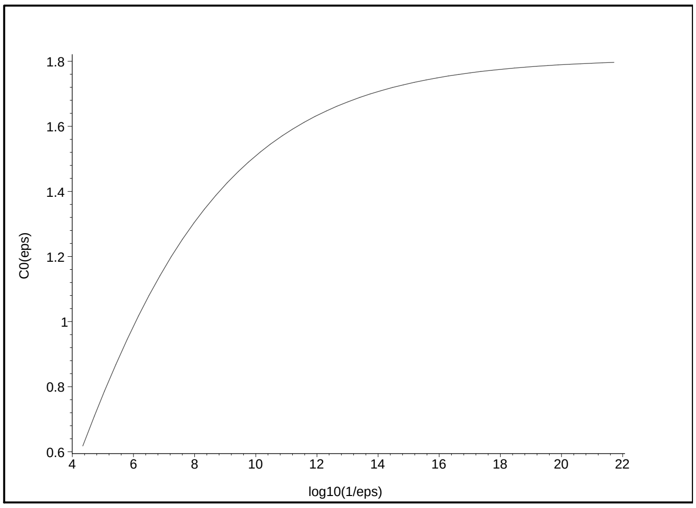

Figure 10: The ε-dependence of the quantity $C_0(\varepsilon)$ of eq.(6.49)

The remaining κ-dependence of $C_0(\varepsilon)$ is negligible in our range($\varepsilon^{\kappa\pi}/\gamma^{1/3} \approx 1$). The order of magnitude of $\partial\phi/\partial R_0$ is correctly reproduced if the integrals are simply cutoff at $\tau = 1/\gamma$ and the exponential term in $k(\theta)$ is ignored. For finite ε the expression in (6.48) should be multiplied by $(1 + O(\varepsilon^s))$ for some small s, which controls the approach of $\partial R/\partial R_0(\theta(\pi/2))$ to A_R^0 as $\varepsilon \to 0$ (see previous section). We compute next (upper bounds to) the corrections to the dominant term and evaluate, for finite ε:

$$\mathcal{I} \equiv \int_0^{\theta(\pi/2)} k(\theta)^2 (R_{L,\varepsilon}(\theta)\frac{\partial R}{\partial R_0}(\theta) - R_{L,f}(\varepsilon)\frac{\partial R}{\partial R_0}(\theta(\pi/2))d\theta \tag{6.50}$$

In Appendix E we show that this integral is bounded and even has a limit, which we denote by D_R^0, as $\varepsilon \to 0$. This is in principle a finite quantity which must be added to the contribution of the "‘rest phase"’ (6.35). However, since $R_{L,\varepsilon}(\theta)$,$\partial R/\partial R_0(\theta)$ are, apart from small oscillations, remarkably constant, the contribution of (6.50) is ≈ 0 The same arguments may be repeated for the partial derivative $\partial\phi/\partial\phi_0$:

[23] $\theta(\pi/2)$ means the θ-value corresponding to $t = \pi/2$; it is $O(1/\varepsilon^{1/2})$

the leading term is $O(1/\gamma^{1/3})$ in analogy to (6.48):

$$\frac{\partial\phi}{\partial\phi_0}(\theta(\pi/2)) = -\frac{7}{12}\frac{\sqrt{3}}{2^{1/3}\gamma^{1/3}}\Gamma(1/3)R_{L,f}(0)A_P^0 \equiv \frac{C_0 A_P^0}{\gamma^{1/3}} \tag{6.51}$$

with A_P of eqn.(6.45). There exists in principle an additional bounded term, which tends to a limit D_P^0 as $\varepsilon \to 0$. It is the sum of $\partial\bar{\phi}/\partial\phi_0$, which approaches (slowly) the value B_P, eq.(6.46) and of the analogon of (6.50), which is ≈ 0. To conclude, the map $(R_0, \phi_0) \Rightarrow (R(\theta(\pi/2)), \phi(\theta(\pi/2)))$ is given by (R_{0L}, ϕ_{0L} are the values of (R_0, ϕ_0) assumed by X_L at τ_0) :

$$R_0 \Rightarrow R_{L,f}(\varepsilon) + A_R(\varepsilon)(R_0 - R_{0L}) + A_P(\varepsilon)(\phi_0 - \phi_{0L}) + higherorders \tag{6.52a}$$

$$\phi_0 \Rightarrow \phi_L(\pi/2,\varepsilon) + \left(-\frac{C_0 A_R(\varepsilon)}{\gamma^{1/3}} + C_R(\varepsilon)\right)(R_0 - R_{0L}) + \left(-\frac{C_0 A_P(\varepsilon)}{\gamma^{1/3}} + C_P(\varepsilon)\right)(\phi_0 - \phi_{0L}) + h.o. \tag{6.52b}$$

where the higher orders are $O(\varepsilon^{2\kappa\pi})$ multiplied by the order of magnitude of the second derivative. The terms $A_R(\varepsilon)$, $A_P(\varepsilon)$ approach the values A_P^0, A_R^0 of eq.(6.45). The terms $C_R(\varepsilon)$, $C_P(\varepsilon)$ are the sums of the corrections to the leading $1/\gamma^{1/3}$ term given by (6.50) (and its analogon for $\partial R/\partial\phi_0$) and the contributions (6.46) of the "‘rest phase"’ $\bar{\phi}$ in (6.47). As $\varepsilon \to 0$ these terms approach (slowly) the limiting values (6.46) given by the $\varepsilon = 0$ equation:

$$C_R(\varepsilon) \Rightarrow B_R^0 + D_R^0 \approx B_R^0 \qquad\qquad C_P(\varepsilon) \Rightarrow B_P^0 + D_P^0 \approx B_P^0 \tag{6.53}$$

Eqns.(6.52a)(6.52b) contain the matrix elements of $\mathbb{DP}_\mathbf{f}$, eqn.(6.20). The complete jacobian $\mathbb{DP}_\mathbb{R}$ evaluated at $(\Delta\eta(0), d\Delta\eta/d\tau(0))$ is computed by performing the matrix multiplication in (6.20). The elements of $\mathbb{DP}_\mathbf{i}$ are in the limit $\varepsilon \to 0$ given by the solutions of eqn.(6.21) with $\gamma = 0$ and η_L replaced by η_{L00} . For reference, for the choice $\tau_0 = 10$ they are:

$$\begin{aligned} \frac{\partial u_{0R}(\tau_0)}{\partial u_{0L}} &= 0.723 & \frac{\partial u_{0R}(\tau_0)}{\partial u\prime_{0L}} &= 1.111 \\ \frac{\partial u\prime_{0R}(\tau_0)}{\partial u_{0L}} &= 0.550 & \frac{\partial u\prime_{0R}(\tau_0)}{\partial u\prime_{0L}} &= 2.230 \end{aligned} \tag{6.54}$$

The mapping $\mathbb{DT}$ for $\tau = \tau_0$ is obtained from (6.14). The dominant terms are:

$$\mathcal{J}_1 \equiv \frac{\partial R}{\partial u_{0L}}(\theta = \pi/2) \cong -0.43, \qquad\qquad \mathcal{J}_2 \equiv \frac{\partial R}{\partial u\prime_{0L}}(\theta = \pi/2) \cong -0.247 \tag{6.55}$$

With this, the right hand side Poincaré map reads (cf.eq.6.16):

$$\begin{aligned} (u_L(0), \frac{du_L}{d\tau}(0)) \Rightarrow (&R_{L,f}(\varepsilon) + \mathcal{J}_1 u_L(0) + \mathcal{J}_2\frac{du_L}{d\tau}(0) + O(\varepsilon^{2\kappa\pi}), \\ &\theta(\pi/2) - \theta(\tau_0) + \phi_L(\pi/2,\varepsilon) - \frac{C_0}{\gamma^{1/3}}(\mathcal{J}_1 u_L(0) + \mathcal{J}_2\frac{du_L}{d\tau}(0)) \\ &+ E_0 u_L(0) + F_0\frac{du_L}{d\tau}(0) + O(\frac{\varepsilon^{2\kappa\pi}}{\gamma^{1/3}}) \end{aligned} \tag{6.56}$$

where $\phi_L(\pi/2,\varepsilon)$ is defined through (6.8a), (6.8b); it contains a "‘secular"’, divergent term $\Phi_{L,R}(\theta(\pi/2))$ described in Lemma 6.2 (cf. eq. (6.10)) and a "‘finite"’ term $\bar{\phi}_L(\pi/2)$ which has a limit $\bar{\phi}_{L,f}(0)$ as $\varepsilon \to 0$, as described in Lemma 6.3. Further, E_0, F_0 approach as $\varepsilon \to 0$ constants obtained from the values in eqns.(6.53),(6.54): $E_0 \approx 1.02$, $F_0 \approx 2.11$. The term $O(\varepsilon^{2\kappa\pi}/\gamma^{1/3})$ is (only qualitatively) justified as follows: the second derivative of the Poincaré map is expected to diverge like $\gamma^{-1/3}$ as $\varepsilon \to 0$, like the phase ϕ_L and its first derivative $\delta\phi$, and is multiplied by terms of $O(u^2) = O(\varepsilon^{2\kappa\pi})$; since - as may already be apparent - we expect bifurcations to occur when $\varepsilon^{\kappa\pi}/\gamma^{1/3}$ is $O(1)$, this correction is of the same order of magnitude as the terms preceding it. However, for small ε, the bifurcation pattern is determined by the divergent terms (see Sections 7 and 8).

7. The Complete Map $\mathbb{P}$ and its Associated Circle Map

7.1. *The map* $\mathbb{P}$

We put now together eqns.(5.52a), (5.52b) and (6.56) to obtain the image of a point

$$P : (w, \frac{dw}{d\theta}) = (\Lambda \cos \Psi_0, -\Lambda \sin \Psi_0)$$

situated in a disk around $(X_L(-\pi/2), dX_L/d\theta_L(-\pi/2))$ under the mapping $\mathbb{P}$ of (3.40) (with the change $t \Rightarrow \theta_{L,R}$ of the independent variable). We may write it as:

$$\begin{aligned} \Lambda \Rightarrow & R_{L,f}(\varepsilon) + \Lambda\varepsilon^{\kappa\pi}\{(\mathcal{J}_1 V_c(0) + \mathcal{J}_2 \frac{dV_c}{d\tau}(0)) \cos(\Psi_0 + \Phi_L(\Lambda) + \Omega) \\ & - (\mathcal{J}_1 V_s(0) + \mathcal{J}_2 \frac{dV_s}{d\tau}(0)) \sin(\Psi_0 + \Phi_L(\Lambda) + \Omega)\} + O(\varepsilon^{\kappa\pi+\delta}) \end{aligned} \tag{7.1a}$$

$$\begin{aligned} \Psi_0 \Rightarrow & \pi + \theta_R(\pi/2) - \theta_R(\tau_0) + \phi_L(\pi/2, \varepsilon) - C_0\Lambda\frac{\varepsilon^{\kappa\pi}}{\gamma^{1/3}}\{(\mathcal{J}_1 V_c(0) + \mathcal{J}_2 \frac{dV_c}{d\tau}(0)) \times \\ & \cos(\Psi_0 + \Phi_L(\Lambda) + \Omega) - (\mathcal{J}_1 V_s(0) + \mathcal{J}_2 \frac{dV_s}{d\tau}(0)) \sin(\Psi_0 + \Phi_L(\Lambda) + \Omega)\} \\ & + \Lambda\varepsilon^{\kappa\pi}\{(EV_c(0) + F\frac{dV_c}{d\tau}(0)) \cos(\Psi_0 + \Phi_L(\Lambda) + \Omega) \\ & - (EV_s(0) + F\frac{dV_s}{d\tau}(0)) \sin(\Psi_0 + \Phi_L(\Lambda) + \Omega)\} + O(\frac{\varepsilon^{2\kappa\pi}}{\gamma^{1/3}}) \end{aligned} \tag{7.1b}$$

where we have used the notation in (6.56).In the region where $\varepsilon^{\kappa\pi}/\gamma^{1/3}$ is $O(1)$ the last two terms in (7.1b) are both of $O(\varepsilon^{\kappa\pi})$ and will be treated together. These expressions are simplified by introducing $\mathcal{M}$ and ξ through:

$$\mathcal{J}_1 V_c(0) + \mathcal{J}_2 \frac{dV_c}{d\tau}(0) \equiv \mathcal{M} \cos \xi \qquad \mathcal{J}_1 V_s(0) + \mathcal{J}_2 \frac{dV_s}{d\tau}(0) \equiv \mathcal{M} \sin \xi \tag{7.2}$$

In eqns.(7.1a),(7.1b) we wrote the Λ-dependence of Φ_L, eq.(5.17) explicitly. The term π in (7.1b) takes care of the minus sign present in the definition of the *half-period* map $\mathcal{P}$ (cf. the definition eq.(3.42)). We perform next the ε,γ-dependent transformation of the angular variable in (7.1a), (7.1b):

$$\Psi_0 = \chi + \pi + \theta_R(\pi/2) - \theta_R(\tau_0) + \phi_L(\pi/2, \varepsilon, \gamma) \equiv S\chi \tag{7.3}$$

with which the transformed mapping $\mathcal{P}$:

$$\mathcal{P} \equiv S^{-1}\mathbb{P}S \tag{7.4}$$

reads:

$$\begin{aligned} \mathcal{P} : \qquad & \Lambda \Rightarrow R_{L,f}(\varepsilon) + O(\varepsilon^{\kappa\pi}) \\ & \chi \Rightarrow \beta\frac{\Lambda}{R_{L,f}(\varepsilon)} \cos(\chi + \tilde{\Sigma}(\varepsilon, \gamma, \Lambda)) + O(\varepsilon^{\kappa\pi}) \end{aligned} \tag{7.5}$$

with the following notations:

$$\beta \equiv C_0\mathcal{M}R_{L,f}(\varepsilon)\frac{\varepsilon^{\kappa\pi}}{\gamma^{1/3}} \tag{7.6}$$

and:

$$\tilde{\Sigma}(\varepsilon, \gamma, \Lambda) \equiv \theta_R(\pi/2) - \theta_R(\tau_0) + \Omega + \phi_{L,R}(\pi/2, \varepsilon) + \xi(\varepsilon, \gamma) + \Phi_L(\varepsilon, \gamma, \Lambda) \tag{7.7}$$

The first terms in eq.(7.7) give (almost) the total θ - variation from $-\pi/2$ to $\pi/2$ (cf.eq.(5.48):

$$\Delta\theta = \theta_R(\pi/2) - \theta_R(\tau_0) + \Omega = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_{-\pi/2}^{\pi/2} |\sin t|^{1/3} dt + \Theta_0(\varepsilon) \tag{7.8}$$

where $\Theta_0(\varepsilon)$ has a finite limit when $\varepsilon \to 0$. For small ε eq.(7.5) shows that, under $\mathcal{P}$, Λ is squeezed to values near $R_{0L}(\varepsilon)$ (as is seen in Figs.8 and 9) and therefore the term $\Phi_L(\varepsilon, \gamma, \Lambda)$ gets very close to $\Phi_L(\varepsilon, \gamma, R_{L,f}(\varepsilon))$. The factor $R_{L,f}(\varepsilon)$ has a limit $R_{L,f}(0)$ as $\varepsilon \to 0$, independent of γ. (cf. Lemma 6.1 and Appendix D). The same is true for the quantity $\mathcal{M}$ (cf.(7.2) according to Lemma 6.4. It is natural then to expect that the *one-dimensional* mapping of the unit circle into itself:

$$\Pi : \qquad \chi \Rightarrow \beta \cos(\chi + \Sigma) \tag{7.9}$$

with

$$\Sigma \equiv \Delta\theta + \phi_L(\pi/2, \varepsilon) + \xi(\varepsilon, \gamma) + \Phi_L(R_{L,f}(\varepsilon), \varepsilon, \gamma) \tag{7.10}$$

contains the essential features of the bifurcation structure of the mapping $\mathcal{P}$, eq.(7.5) and thus of $\mathbb{P}$. In the following subsection we present some relevant features of this mapping, which is otherwise well studied [Zeng & Glass, 1989][Collet & Eckmann, 1983](this is the standard reference on one-dimensional mappings; however, the map (7.9) falls a little outside the class of maps considered there.).In the next section, we discuss its relation to the real Poincaré mapping of the Duffing equation $\mathcal{P}$ of (7.5).

Clearly, the mapping Π may show a bifurcation structure in the region of parameter space where $\beta = O(1)$. This justifies some of the statements made before concerning the orders of magnitude coming into play (see comments following eq.(7.1b) and eq.(6.56) above). Referring to the discussion of Section 6.3, especially to that accompanying Figs.8 and 9, it is easy to give a "‘physical"’ rationale for the parameter β: in a τ-time interval of order $1/\gamma$ the motion is (almost) hamiltonian and the original disk (ellipse) in the $(u(0), du/d\tau(0)$ plane with radius of $O(\varepsilon^{\kappa\pi})$ is stretched into an increasingly thin filament of increasing angular aperture wrapping itself around a circle of radius $R_{Lf}(\varepsilon)$. The angular aperture is increasing at the rate $\theta^{1/4}$ (cf.eq.(6.32)), where:

$$\theta \approx \sqrt{3} \int^{\tau} \tau\prime^{1/3} d\tau\prime \approx \frac{3\sqrt{3}}{4} \tau^{4/3}$$

In terms of θ the τ-time $1/\gamma$ is $O(1/\gamma^{4/3})$ so that the original aperture of $O(\varepsilon^{\kappa\pi})$ becomes in a $\tau-$time $1/\gamma$ of $O(\varepsilon^{\kappa\pi} \times 1/\gamma^{1/3})$ which is precisely the order of magnitude of β in (7.7). At τ-times larger than $1/\gamma$ the angular aperture does not increase any more considerably, but the area of the "‘wrapped"’ rectangle decreases simply due to the damping by a factor $\approx \varepsilon^{\kappa\pi}$.

7.2. *The circle map* Π

In this section we gather some properties of the map Π of eqn.(7.9).Clearly its features are periodic in Σ. From the definition of the latter in eqn.(7.10) the dominant term for small ε is $\Delta\theta$ of (7.8) which behaves like $1/\sqrt{\varepsilon}$, i.e. like $\Gamma^{1/3}$. Thus, we expect the bifurcation pattern in the $\Gamma - \Delta$ plane to have at high forcing an increasingly better periodicity in $\Gamma^{1/3}$. At fixed ε, β decreases with increasing γ, i.e. with increasing Δ. For a comparison with "‘normal"’ bifurcation plots, we draw bifurcation lines in a $\Sigma - (-\beta)$ plane.

(i)For $\beta < 1$ (high damping), the equation:

$$\beta \cos(\chi + \Sigma) = \chi \tag{7.11}$$

has, for all Σ, only one solution χ_s. This solution is a *stable* fixed point of Π since $|d\Pi/d\chi(\chi_s)| < 1$. Even more,

Lemma 7.1 *The solution χ_s of (7.11) is the only invariant set under Π if $\beta < 1$*

Indeed, $|d\Pi/d\chi| < 1$ for all χ, so that the distance between any two χ_1, χ_2 is contracted under Π if $\beta < 1$:

$$|\Pi(\chi_2) - \Pi(\chi_1)| \leq \sup_{\chi \in (\chi_1, \chi_2)} |\Pi\prime(\chi)||(\chi_2 - \chi_1)| \leq |(\chi_2 - \chi_1)| \tag{7.12}$$

Thus, the sequence of all iterates of any χ under Π converges (to χ_s). (ii)If $\beta < \pi$, Π maps the interval $(-\pi, \pi)$ into itself, so that the theory of iterated mappings of intervals, as presented in Collet & Eckmann [1983] and Guckenheimer & Holmes [1983] may be directly taken over. If $\Sigma = 0$, the mappings $\Pi(\beta, 0)$ are unimodal in the sense of Collet & Eckmann [1983]. The mappings $\Pi(\beta, \Sigma = -\pi/2)$ apply $[0, \pi]$,$[-\pi, 0]$ into themselves and - if restricted to these intervals - make up a full family of unimodal maps[Collet & Eckmann, 1983, §III.1,p.174], for $0 < \beta < \pi$. Moreover, for all values of Σ, the functions $\Pi(\beta, \Sigma)(\chi)$ have a negative Schwarz derivative:

$$Sf \equiv \frac{f\prime\prime\prime}{f\prime} - \frac{3}{2}\left(\frac{f\prime\prime}{f\prime}\right)^2 < 0 \tag{7.13}$$

For unimodal families with negative Schwarz derivative there exists a sequence of values $\beta_1 < \beta_2...$ which accumulates at a value $\beta_c < \pi$ and for which there exist *superstable* orbits of period 2^p. For $\beta = \beta_c$ there exists a (nonperiodic) attracting Cantor set for the action of $\Pi(\beta, -\pi/2)$. For larger values the motion may be "‘chaotic"’ (with *sensitive dependence on the initial conditions*). Thus, we expect that chaotic motion occurs in the Duffing equation before (i.e. at smaller β) the Poincaré map covers the whole angular range $0 < \phi < 2\pi$ (cf.Fig.9). Even without the restriction to unimodal maps,i.e. for arbitrary choices of Σ,condition (7.13)places restrictions on the possible invariant sets of Π (see Lemma 7.2 below). (iii)At some fixed values of Σ, if we increase β, we reach a value beyond which eqn.(7.11) admits of three (or more) solutions. The limiting values β_S are those for which the line $\Pi = \chi$ is tangent to the graph of $\Pi(\chi)$, i.e.

$$\begin{aligned} \chi_S &= \beta_S \cos(\chi_s + \Sigma) \\ 1 &= -\beta_S \sin(\chi_s + \Sigma) \end{aligned} \tag{7.14}$$

It follows that:

$$\beta_S^2 = 1 + \chi_S^2 \geq 1 \tag{7.15}$$

and thus $\beta_S = 1$ only if χ_S=0; eqns.(7.14) imply then $\Sigma = -\pi/2 \pmod{2\pi}$. The bifurcations occuring when $\beta = \beta_S$ are - if $\Sigma \neq -\pi/2$ - of *saddle-node* type : at neighboring larger values of β, two more solutions appear,corresponding to a stable and an unstable orbit of period 1 (under the action of Π). If $\Sigma = -\pi/2$, the bifurcation at $\beta = 1$ is of the *pitchfork* type: the unique solution existing at $\beta < 1$ loses its stability and a pair of stable solutions of period 1 appear at $\beta > 1$.

(iv) From (7.14) one can obtain the exact form of the bifurcation line $\beta = \beta_S(\Sigma)$.Near $\beta = 1$ it has a cusp: indeed, let in eq.(7.14) $\Sigma = -\pi/2 + \sigma$ so that (7.14) implies:

$$\tan(\chi_S + \sigma) = \chi_S \tag{7.16}$$

For small χ_S and σ this means, using (7.15):

$$\chi_S(\sigma) \approx (3\sigma)^{1/3} \qquad \beta_S(\sigma) \approx 1 + (3\sigma)^{2/3} \tag{7.17}$$

which shows the cusp behaviour. Fig.11 shows the saddle-node bifurcation lines in a $\Sigma - (-\beta)$ plane (with an origin for Σ defined $mod(2\pi)$.

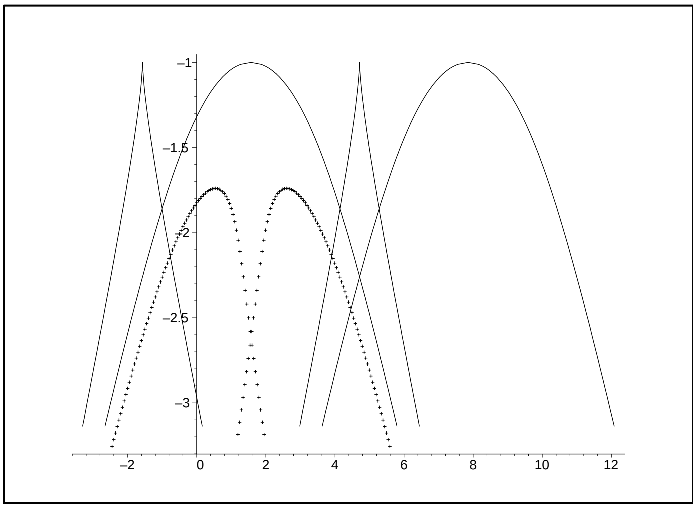

Figure 11: The bifurcation lines in the $\Sigma - (-\beta)$ plane

(v)If at a fixed point χ_F of $\Pi(\chi)$, $d\Pi/d\chi(\chi_F) = -1$, the map Π has a *flip* bifurcation[24], if certain transversality conditions are obeyed. In this case, these latter[25] are at fixed Σ simply: $\chi_F^2 + 2 \neq 0$, $\chi_F^2 + 2/3 \neq 0$. The analogon of (7.14) is now:

$$\begin{aligned} \chi_F &= \beta_F \cos(\chi_F + \Sigma) \\ 1 &= \beta_F \sin(\chi_s + \Sigma) \end{aligned} \tag{7.18}$$

from which one sees that $\beta_F(\Sigma) \geq 1$, with equality only if $\chi_F = 0$, which occurs at $\Sigma = \pi/2$.

(vi)Eqn.(7.18) allows an exact determination of the flip bifurcation lines, see Fig.11. Their maximum at the line $\beta = 1$ is quadratic, which may be easily seen as follows: let $\Sigma = \pi/2 + \sigma$ so that (7.18) implies:

$$\tan(\chi_F + \sigma) = -\chi_F \tag{7.19}$$

from which, for small χ_F, σ, one deduces:

$$\chi_F(\sigma) = -\frac{\sigma}{2} + O(\sigma^3). \tag{7.20}$$

Then (7.18) leads to (cf.(7.15)):

$$\beta_F(\sigma) \approx 1 + \frac{\sigma^2}{8} \tag{7.21}$$

(vii)The flip bifurcation curves are much broader than the saddle-node ones (see Fig. 11). We give a simple estimate of the ratio of their widths (measured at the points where they intersect with the smallest values of β): at such points with coordinates (Σ^0, β^0), the four equations (7.14), (7.18) (for the four unknowns Σ^0, β^0, χ_S^0, χ_F^0) imply:

$$\chi_F^0 = \pm\chi_S^0 = \pm\sqrt{(\beta^0)^2 - 1} \tag{7.22}$$

We look for solutions with $\beta^0 < \pi$, which means that $|\chi_F^0|, |\chi_S^0| < \pi$. The second of each pair of eqns. (7.14),(7.18) exclude in turn the possibility $\chi_F^0 = \chi_S^0$. From each of the pairs (7.14),(7.18) one deduces that:

$$\cot(\chi_F^0 + \Sigma^0) = \chi_F^0 = -\chi_S^0 = \cot(\chi_S^0 + \Sigma^0) \tag{7.23}$$

[24] see Guckenheimer & Holmes [1983, ch.III,Theorem 3.5.1]
[25] Conditions (F1) and (F2) in Guckenheimer & Holmes [1983]

so that:

$$\chi_F^0 = \chi_S^0 (mod \pi)$$

With (7.22) this is possible only if $\chi_S^0 = \pm\pi/2$, $\chi_F^0 = \mp\pi/2$. If $\chi_S^0 = \pi/2$,(7.23) and (7.14) imply $\Sigma^0 = -\arctan(\pi/2)$ (mod 2π). If $\chi_S^0 = \pi/2$, the solution accepted by (7.14) is $\Sigma^0 = \pi + \arctan(\pi/2)$ (mod 2π).The ratio ρ of the length of two consecutive intervals between possible values of Σ^0 is:

$$\rho = \frac{\pi - 2\arctan(\pi/2)}{\pi + 2\arctan(\pi/2)} \approx 0.22 \tag{7.24}$$

(viii)At a larger value of β, $\beta_2(\Sigma)$ the orbits of period two which appeared at the flip bifurcation described above undergo another period doubling bifurcation. Fig.11 shows the results of a numerical calculation of $\beta_2(\Sigma)$. As is well known in many instances, one can find values $\beta_4(\Sigma)$, $\beta_8(\Sigma)$ at which further bifurcations to periodic stable solutions of period 4,8..etc. occur. For $\Sigma = 0$, one obtains $\beta_2 = 1.8271$, $\beta_4(0) = 1.9429$, $\beta_8(0) = 1.9674$...

(ix)For a complete description of the situation, one needs also an argument that, at least for values of β not too large, the mapping Π contains no other invariant sets apart from the fixed points (or orbits of period 2,etc.) described above. Because the family $\Pi(\beta, \Sigma)$ does not fall entirely under the classes of one-dimensional mappings described in Collet & Eckmann [1983] and Guckenheimer & Holmes [1983], we give a statement which guarantees the (expected) absence of supplementary invariant sets for small enough β:

Lemma 7.2 *Let $\beta_{2u}(\Sigma)$ be the second positive root (i.e.different from $-\Sigma$) of the equation:*

$$\Pi(\beta, \Sigma; \chi = \beta) = \beta\cos(\beta + \Sigma) = -\Sigma \tag{7.25}$$

if $\Sigma < 0$ and the positive root of (7.25) if $\Sigma > 0$ $(-\pi < \Sigma < \pi)$. Let $\beta_{2d}(\Sigma)$ be the second positive root (i.e.different from $\pi + \Sigma$) of :

$$\Pi(\beta, \Sigma; \chi = -\beta) = \beta\cos(-\beta + \Sigma) = -\pi - \Sigma \tag{7.26}$$

if $\Sigma < 0$ and the first positive root of

$$\Pi(\beta, \Sigma; \chi = -\beta) = \beta\cos(-\beta + \Sigma) = \pi - \Sigma \tag{7.27}$$

if $\Sigma > 0$. Let:

$$\beta_e(\Sigma) = \min[\pi, \beta_{2u}(\Sigma), \beta_{2d}(\Sigma)] \tag{7.28}$$

Then, for $0 < \beta < \beta_e(\Sigma)$ the invariant sets of $\Pi(\beta, \Sigma)$ consist of at most three fixed points and two orbits of period two.

The proof of this statement is relegated to Appendix F, because it is not very short; it uses the property of Π to have a negative Schwarz derivative (7.13) and owes a lot to the presentation in ch.III of Collet & Eckmann [1983]. The conditions (7.25),(7.26),(7.27) describe *superstable* orbits of period two (i.e. orbits which pass through the maximum β or the minimum $-\beta$ of $\Pi(\beta, \Sigma; \chi)$).If,e.g. $\Sigma < 0$, the maximum and minimum of Π occur at $\chi_M = -\Sigma$, $\chi_m = -\pi - \Sigma$ in turn, $\Pi(\chi_M) = \beta$, $\Pi(\chi_m) = -\beta$: eq.(7.25) states that the iteration of χ_m under Π should repeat itself after two steps. For the root $\beta_{1u} \equiv -\Sigma$ of (7.25) the fixed point of Π lies on the maximum of $\Pi(\chi)$ (and on the minimum for $\beta_{1d} \equiv \pi + \Sigma$ in (7.26).

This closes the qualitative discussion of the mapping Π.

8. The Bifurcations of Periodic Solutions at Large $\Gamma(\varepsilon \to 0)$

8.1. *The bifurcations of* Π *in the* $\Gamma - \Delta$ *plane*

Instead of the parameters Γ,Δ, we can use $\beta(\varepsilon, \gamma)$, $\Sigma(\varepsilon, \gamma)$ of eqns.(7.7),(7.10).If we believe that the map (7.9) reproduces the main features of the Poincaré map $\mathbb{P}$, then the tips of the bifurcation curves lie - alternatively saddle-nodes and odd-periodic - simply periodic(flip) - along the line $\beta = 1$. Above certain critical values of β - we denote them by $\hat{\beta}$ - there appear bifurcations to orbits with higher period. Assuming ε is so small that $R_{0L}(\varepsilon)$, $\mathcal{M}(\varepsilon)$ may be replaced with their limiting values at $\varepsilon = 0$ (from eqns. (5.53) and (6.55) $\mathcal{M}_0 = 0.6814$), we obtain for the asymptotic form of the bifurcation lines of (7.9):

$$\Delta_c(\Gamma) = \frac{1}{12\pi} \ln \Gamma - \frac{1}{3\pi} \ln \ln \Gamma - \frac{1}{\pi} \ln \frac{\hat{\beta}}{\mathcal{M}_0 R_{L,f}(0) C_0} + O(\frac{\ln \ln \Gamma}{\ln \Gamma}) \tag{8.1}$$

with $R_{L,f}(0)$ of Lemma 6.1 and C_0 of eq.(6.48). Fig.12 gives an idea of the appearance of the lines $\beta = \hat{\beta}$ in a $\Delta/\ln\Gamma$ vs. $\ln\Gamma$ plot, with $\hat{\beta} = 0.8$,1(solid) and 2 and of the asymptotic approximation (dotted) of (8.1). The bifurcation structure of Π, eq.(7.9) is periodic in Σ, i.e. the bifurcation pattern

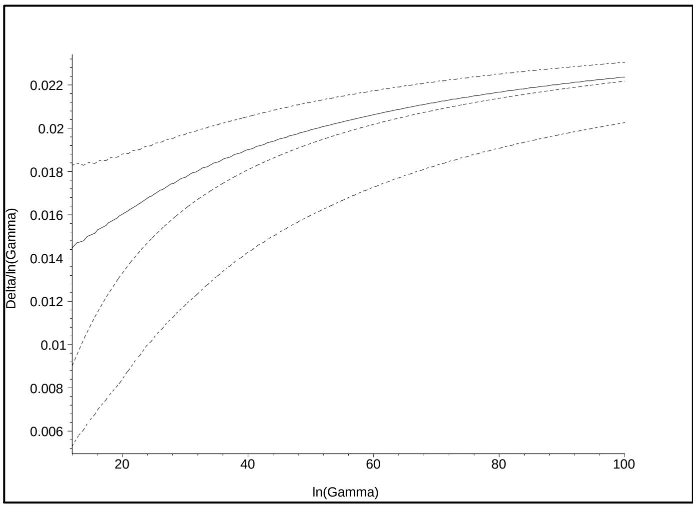

Figure 12: The lines β=0.8,1(solid) and 2 and the approximation (8.1) to $\beta = 1$,(dots)

repeats itself along lines of fixed κ (eq.(1.21) in intervals $\Delta\Gamma$ obeying:

$$\Sigma(\Gamma + \Delta\Gamma, \kappa \ln(\Gamma + \Delta\Gamma)) - \Sigma(\Gamma, \kappa \ln \Gamma) = 2\pi \tag{8.2}$$

The dominant term in Σ, eq.(7.10) is $\Delta\theta$, eq.(7.8), which is proportional to $1/\sqrt{\varepsilon}$, i.e. to $\Gamma^{1/3}$ (cf.eq.(1.7). If $1/\sqrt{\varepsilon}$ is much larger than $\gamma^{-1/3}$ - the magnitude of the second term $\phi_{L,R}(\pi/2, \varepsilon)$ in (7.10)- the pattern repeats itself in equal intervals of $\Gamma^{1/3}$, independently of κ. The period is:

$$\Delta(\Gamma^{1/3}) = \frac{2\pi}{\sqrt{3}} \frac{1}{\int_{-\pi/2}^{\pi/2} |\sin t|^{1/3}} dt \cong 2.804 \tag{8.3}$$

Since the maxima of the saddle-node and flip bifurcation lines occur at $\Sigma = -\pi/2, \pi/2$ $(mod 2\pi)$, they are asymptotically equidistant in $\Gamma^{1/3}$ as shown in eq.(1.6). At smaller values of κ (or smaller values of ε) periodicity in $\Gamma^{1/3}$ is still the dominant feature but the shape of the bifurcation lines is distorted. Fig.13 shows the appearance of the lines $\Sigma(\varepsilon, \gamma) = const$ in a $\Delta/\ln\Gamma$ vs. $\Gamma^{1/3}$ plot (the horizontal line is $\beta(\varepsilon, \gamma) = 1$). We now show that near $\beta = 1$ the bifurcation structure of $\mathbb{P}$ must reproduce the one of Π if Γ is sufficiently large (ε sufficiently small).

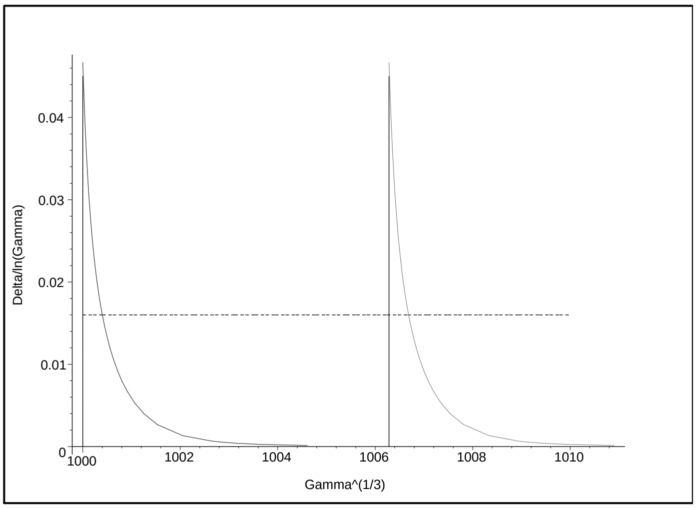

Figure 13: The lines $\Sigma = const$ after inclusion of the other terms in (7.10)

8.2. *Inferences from* Π *to* $\mathbb{P}$

The mapping $\mathcal{P}$, eq.(7.5) is equivalent to $\mathbb{P}$ and may be written in components $(\mathcal{P}_\Lambda, \mathcal{P}_\chi)$:

$$\Lambda \Rightarrow R_{L,f}(\varepsilon) + \varepsilon^{\kappa\pi} G(\Lambda, \chi, \varepsilon, \gamma) \equiv \mathcal{P}_\Lambda(\Lambda, \chi, \varepsilon, \gamma) \tag{8.4a}$$

$$\chi \Rightarrow \beta(\varepsilon, \gamma) \frac{\Lambda}{R_{L,f}(\varepsilon)} \cos(\chi + \tilde{\Sigma}(\lambda, \varepsilon, \gamma)) + \varepsilon^{\kappa\pi} H(\Lambda, \chi, \varepsilon, \gamma) \equiv \mathcal{P}_\chi(\Lambda, \chi, \varepsilon, \gamma) \tag{8.4b}$$

$$\tilde{\Sigma}(\Lambda, \varepsilon, \gamma) \equiv \Sigma(\varepsilon, \gamma) + \Phi_L(\Lambda, \varepsilon, \gamma) - \Phi_L(R_{L,f}(\varepsilon), \varepsilon, \gamma)) \tag{8.4c}$$

where G and H are differentiable functions of Λ and χ, bounded and with bounded derivatives (up to the third order) with respect to Λ,χ as $\varepsilon \to 0$ (uniformly with respect to κ, as long as $\kappa > \kappa_0 > 0$). The difference $\Phi_L(R_{L,f}) - \Phi_L(\Lambda)$ in (8.4c) is proportional to $\Lambda^2 - R_{L,f}^2$ and thus, using (8.4a), to $\varepsilon^{\kappa\pi}$. It is convenient to regard in (8.4a), (8.4b) the quantity $\varepsilon^{\kappa\pi}$ multiplying the functions G and H as an independent parameter q which may be set equal to zero (and obtain thus the mapping Π, eq.(7.9) or to $\varepsilon^{\kappa\pi}$ (to obtain $\mathcal{P}$). With the assumptions concerning the functions G and H in (8.4a), (8.4b) the approach of $\mathcal{P}$ to Π as $q \to 0$ is uniform with respect to $\chi, \Lambda, \beta \in [0, 2\pi] \times [0, \Lambda_M] \times [\beta_-, \beta_+]$, where Λ_M is an upper bound on Λ, as derived in Section 4.1 and $\beta_- < 1$, $\beta_+ > 1$ surround $\beta = 1$. As a consequence, we shall show generally that the properties of the circle map Π are carried over into those of the complete mapping $\mathcal{P}$ for q sufficiently small, at fixed ε; we may replace then the phrase "'as $q \to 0$"' with the one "'as $\varepsilon \to 0$"' using the approximate periodicity in Σ of $\mathcal{P}_\chi$, eqn.(8.4b. Indeed, if some property holds "'for $q < q_0$"', it will be true for all $\varepsilon < q_0^{1/(\kappa\pi)}$. In the process of letting $q \to 0$ and choosing a correspondingly small ε, we assume first $\beta = const$. With this, it is easy to show:

Lemma 8.1 *If χ_0 is a fixed point of Π ,eq.(7.9), with $d\Pi/d\chi(\chi_0) \neq 1$, then, for sufficiently small ε, $\mathcal{P}$ also has a fixed point $(\tilde{\Lambda}_0, \tilde{\chi}_0)$ so that:*

$$|\tilde{\Lambda}_0 - R_{L,f}|, |\tilde{\chi}_0 - \chi_0| = O(\varepsilon^{\kappa\pi}) \tag{8.5}$$

The argument - essentially the same as for the implicit function theorem - uses the periodicity of Π with respect to Σ and the fact that $\mathcal{P}_\chi$ tends to Π as $\varepsilon \to 0$. First, for any χ the equation $\mathcal{P}_\Lambda(\Lambda, \chi, q) = \Lambda$ has a solution $\Lambda = \Lambda(\chi, q)$ contained in $(R_{L,f}(\varepsilon) - qM, R_{L,f}(\varepsilon) + qM)$ where M is an upper bound for $|G(\Lambda, \chi)|$ of (8.4a). Indeed, the function $\mathcal{P}_\Lambda(\Lambda, \chi) - \Lambda$ is monotonical with respect to Λ for small q and assumes opposite signs at the ends of the interval. The function $\Lambda(\chi, q)$ is differentiable and

$$\frac{d\Lambda}{d\chi} = -\frac{\partial \mathcal{P}_\Lambda/\partial\chi}{\partial \mathcal{P}_\Lambda/\partial\Lambda} = O(q) \tag{8.6}$$

Let now $\delta > 0$ be such that: $|d\Pi/d\chi(\chi_0) - 1| > \delta$ and χ_-, χ_+ so that $\chi_0 \in (\chi_-, \chi_+)$,

$$|\Pi(\chi_{+/-}) - \chi_{+/-}| > \delta \tag{8.7}$$

and

$$|\frac{d\Pi}{d\chi}(\chi) - 1| > \delta, \qquad \chi \in (\chi_-, \chi_+). \tag{8.8}$$

Using in $\mathcal{P}_\chi(\Lambda, \chi, q)$ the function $\Lambda(\chi, q)$ of (8.6), we evaluate (cf.(8.4c):

$$\begin{aligned}|\mathcal{P}_\chi(\chi, \Lambda(\chi)) - \Pi(\chi)| &< |\beta(\frac{\Lambda}{R_{L,f}(\varepsilon)} - 1)\cos(\chi + \tilde{\Sigma}) vert \\ &+ |\beta\cos(\chi + \tilde{\Sigma}) - \beta\cos(\chi + \Sigma)| + O(q).\end{aligned} \tag{8.9}$$

Both differences appearing on the right hand side of (8.9) may be made as small as one wishes by allowing q to be sufficiently small. In particular, the left hand side may become less than $\delta/2$ with a choice of q valid uniformly with respect to κ, provided $\kappa > \kappa_0 > 0$ and with respect to $\chi \in [0, 2\pi]$. Indeed, one estimates (cf.eq.(8.4c)):

$$\Phi_L(\Lambda) - \Phi_L(R_{L,f}(\varepsilon)) < const \times (\Lambda^2 - R_{L,f}^2)\frac{\beta}{R_{L,f}\mathcal{M}}\frac{1}{\kappa^{2/3}(\ln(1/\varepsilon))^{2/3})} = O(\frac{q}{\kappa_0^{2/3}}) \tag{8.10}$$

We conclude from (8.7) and (8.9) that the function $\mathcal{P}_\chi(\Lambda(\chi), \chi, q) - \chi$ changes sign between χ_- and χ_+. It is also monotonical there, if q is small enough. Indeed,the difference

$$\begin{aligned}|\frac{d\mathcal{P}_\chi}{d\chi} - \frac{d\Pi}{d\chi}| = &\left|\frac{d\Lambda}{d\chi}\right|\left|\frac{\mathcal{P}}{\Lambda} - \beta\frac{\Lambda}{R_{L,f}(\varepsilon)}\sin(\chi + \tilde{\Sigma}(\Lambda))\frac{d\Phi_L}{d\Lambda}\right| \\ &+ \beta\frac{\Lambda}{R_{L,f}(\varepsilon)}\left|\sin(\chi + \tilde{\Sigma}(\Lambda)) - \sin(\chi + \Sigma(\Lambda))\right| + O(q)\end{aligned} \tag{8.11}$$

may be rendered as small as one wishes, upon using (8.6),(8.10), in particular smaller than $|1 - d\Pi/d\chi(\chi)|$, cf.(8.8). Thus for q small enough, $|d\mathcal{P}_\chi/d\chi - 1| \neq 0$. It follows that, as announced, $\mathcal{P}_\chi(\Lambda, \chi) - \chi$ vanishes just once between χ_- and χ_+. This proves Lemma 8.1.

Assume now $\Pi(\chi)$ has several fixed points with χ_i with $d\Pi/d\chi(\chi_i) \neq 1$. Choosing for every χ_i corresponding intervals (χ_-, χ_+) as in (8.7), we may assume - using the uniformity of the approach of $\mathcal{P}_\chi$ to Π - that (8.7) is valid at all points of $[0, 2\pi]$ which lie outside the union of these intervals. We call this union I_δ. Then, by letting q be small enough, we may ensure that:

$$|\mathcal{P}_\chi(\Lambda, \chi) - \chi| > |\Pi(\chi) - \chi| - |\mathcal{P}_\chi(\Lambda, \chi) - \Pi(\chi)| > \delta/2, \qquad (\Lambda, \chi) \in [0, \Lambda_M] \times \mathbb{C}I_\delta \tag{8.12}$$

Further,

$$|\mathcal{P}_\Lambda(\Lambda, \chi) - \Lambda| > 0, \qquad \chi \in I_\delta, \quad \Lambda \notin (R_{L,f} - qM, R_{L,f} + qM) \tag{8.13}$$

as follows from the definition of M. But in each component of the remaining domain $(R_{L,f} - qM, R_{L,f} + qM) \times I_\delta$,according to the argument of Lemma 8.1 above, the equality :

$$|\mathcal{P}_\Lambda(\Lambda, \chi) - \Lambda| + |\mathcal{P}_\chi(\Lambda, \chi) - \chi| = 0$$

holds at only one point (Λ, χ), which is the "‘evolution"’ with q of the fixed point of the circle map Π. We conclude thus:

Lemma 8.2 *For ε sufficiently small, the fixed points of the complete mapping $\mathcal{P}$ are in one-to-one correspondence with the fixed points χ_i of the circle map Π, provided the latter are such that $|d\Pi/d\chi(\chi_i) - 1| > \delta$, for some $\delta > 0$.*

Since only continuity arguments are involved, the same result is true for iterates of any order p of $\mathcal{P}$, when compared to the iterates Π^p of the circle map. Also, if $\Pi(\chi)$ has an attracting/repelling fixed point with an eigenvalue of the linear part sufficiently far from unity, the eigenvalue of the linearization of $\mathcal{P}$ around the corresponding fixed point is also smaller/larger than unity. It is easy to ascertain the stability of eigenvalues of Π equal to unity (saddle-node bifurcations) in the transition to the complete mapping $\mathcal{P}$, if one allows for a further degree of freedom: we take it to be the variable β of (7.6)[26]. When varying β, we assume the quantity Σ is held fixed. This is only approximately the same as holding ε fixed: the departure is larger at smaller damping (see Fig.13). One can state:

Lemma 8.3 *Let $2\pi - \delta > |\Sigma + \pi/2| > \delta > 0$ and χ_0, β_0 so that $\Pi(\chi) \equiv \beta_0 \Pi^0(\chi)$ (cf.eq.(7.9), $\Pi^0(\chi) \equiv \cos(\chi + \Sigma)$) obeys:*

$$\beta_0 \Pi^0(\chi_0) = \chi_0, \qquad \beta_0 \frac{\partial \Pi^0}{\partial \chi}(\chi_0) = 1 \tag{8.14}$$

Then, for q small enough, the set of equations

$$\mathcal{P}_\Lambda(\Lambda, \chi, \beta, q) = \Lambda, \qquad \mathcal{P}_\chi(\Lambda, \chi, \beta, q) = \chi, \qquad \det|\mathbb{I} - \mathbb{D}\mathcal{P}| = 0 \tag{8.15}$$

has a unique solution $(\Lambda(q), \chi(q), \beta(q))$ which tends to $(R_{L,f}, \chi_0, \beta_0)$ as $q \to 0$.

The argument is almost the same as in Lemma 8.1. In view of the continuity of $\mathcal{P}_\chi$ and its derivatives as $q \to 0$ it is true that: for any $e > 0$ we can find q_0 so that at fixed Σ, for $0 < q < q_0$

$$|\mathcal{P}_\Lambda(\Lambda, \chi, \beta, q) - R_{L,f}| + |\mathcal{P}_\chi(\Lambda, \chi, \beta, q) - \beta \Pi^0(\chi)| < e \tag{8.16}$$

$$|\frac{\partial \Pi_\Lambda}{\partial \Lambda}| + |\frac{\partial \Pi_\Lambda}{\partial \chi}| + |\frac{\partial \Pi_\Lambda}{\partial \beta}| < e \tag{8.17}$$

and

$$|\frac{\partial \mathcal{P}_\chi}{\partial \chi} - \beta \frac{\partial \Pi^0}{\partial \chi}| < e, \qquad |\frac{\partial^2 \mathcal{P}_\chi}{\partial \chi^2} - \beta \frac{\partial^2 \Pi^0}{\partial \chi^2}| < e \tag{8.18}$$

for $0 < \Lambda < \Lambda_M, \beta_- < \beta < \beta_+$ and all χ. Further, the condition on Σ in the statement of the Lemma ensures that χ_0 stays away from 0, i.e. $\Pi(\chi) > d > 0$ in a domain $(\chi,\chi_+) \times (\beta_-, \beta_+)$ containing (χ_0, β_0),for some $d > 0$. If $e < d/2$, it follows from (8.16) that $|\mathcal{P}_{\Lambda,\chi,\beta,q}| > d/2 > 0$ on the same interval in χ and β. As in Lemma 8.1, the first equation (8.15) determines for small q and all $\chi \in [0, 2\pi]$ a unique solution $\Lambda(\chi, \beta, q)$, with $\partial\Lambda/\partial\chi$, $\partial\Lambda/\partial\beta$ of $O(q)$ and $|\Lambda(\chi, \beta, q) - R_{L,f}| = O(q)$. With the notation (similar to Π^0 in (8.14), cf.eq.(8.4b))

$$\mathcal{P}_\chi \equiv \beta \mathcal{P}^0_\chi + qH(\Lambda, \chi, \beta, \Sigma) \tag{8.19}$$

the second equation (8.15) is:

$$\beta - \frac{\chi - qH(\Lambda(\chi, \beta, q), \chi, \beta, \Sigma)}{\mathcal{P}^0_\chi(\chi, \Lambda(\chi, \beta, q), q, \beta)} = 0 \tag{8.20}$$

In (8.20),$\mathcal{P}^0_\chi$ depends on β also through the variable $\tilde{\Sigma}$ (cf.eqs.(8.10) and (8.4b)); from (8.10) it follows that $\partial\tilde{\Sigma}/\partial\beta$ is $O(\Lambda - R_{L,f})$, i.e. $O(q)$. Since $|\mathcal{P}_\chi| > d/2 > 0$ and $\beta > \beta_-$, it follows that the partial derivative of the left hand side of (8.20) with respect to β is $1 - O(q)$ so that (8.20) determines a

[26]The ε-dependence of the factors $R_{L,f}, \mathcal{M}$ in (7.7) is very mild and may be overlooked for all purposes; β is essentially $\varepsilon^{\kappa\pi}/\gamma^{1/3}$

function $\beta(\chi, q)$ for $\chi_- < \chi < \chi_+$, with $\partial\beta/\partial\chi = O(1)$. The third equation (8.15) may be written after rearrangements:

$$\beta\left[\frac{\partial\mathcal{P}^0_\chi}{\partial\chi} - \left(\frac{\partial\mathcal{P}^0_\chi}{\partial\chi}\frac{\partial\mathcal{P}_\Lambda}{\partial\Lambda} - \frac{\partial\mathcal{P}^0_\chi}{\partial\Lambda}\frac{\partial\mathcal{P}_\Lambda}{\partial\chi}\right)\right] = 1 - q\left[\frac{\partial H}{\partial\chi}\frac{\partial\mathcal{P}_\Lambda}{\partial\Lambda} - \frac{\partial H}{\partial\Lambda}\frac{\partial\mathcal{P}_\Lambda}{\partial\chi}\right] \tag{8.21}$$

where we use $\beta = \beta(\chi, q)$. Since the derivatives of $\mathcal{P}_\Lambda$ in (8.21) are $O(q)$, the latter may be rewritten as:

$$\beta\left[\frac{\partial\mathcal{P}^0_\chi}{\partial\chi} + O(q)\right] = 1 + O(q) \tag{8.22}$$

From (8.20) and (8.22) we deduce:

$$\chi\frac{\partial\mathcal{P}^0_\chi}{\partial\chi} - \mathcal{P}^0_\chi - O(q) = 0 \tag{8.23}$$

Now, at $q = 0$ eq.(8.23) has the solution $\chi = \chi_0$. This solution is a simple zero of the combination $\chi\partial\Pi^0/\partial\chi - \Pi^0$ because its derivative $\chi\partial^2\Pi^0/\partial\chi^2$ is nonvanishing at χ_0. This is a consequence of $\partial^2\Pi^0/\partial\chi^2 = -\Pi^0$ and of the condition on Σ. Therefore, this combination is monotonical on an interval (χ_-, χ_+) around χ_0, possibly included in the former, and acquires at the ends absolute values larger than some $d > 0$, possibly smaller than the former. Choosing again $e < d/2$ we can find q_0 so that, for $q < q_0$ (i)the right hand side of (8.23) has opposite signs at χ_-, χ_+ and (ii)its derivative has a constant sign on (χ_-, χ_+) (as a consequence of the second equation in (8.18)). Thus there exists only one solution $\chi(q)$ for every q sufficiently small and it approaches χ_0 as $q \to 0$. This leads then to solutions $\Lambda(\chi(q))$, $\beta(\chi(q))$ with the properties announced in Lemma 8.3 and ends the argument.

Since now β changes with q, one may wonder about the values of the forcing Γ (or ε) above (below) which we may set $\varepsilon^{\kappa\pi} = q$ (and thus replace "'small enough q"' with "'small enough ε"'. According to Lemma 8.3, for $q < q_0$, $\beta(q)$ is contained in an interval $[\beta_-, \beta_+]$. For such values of β, the solutions $\kappa(\varepsilon, \beta)$ of the equation $\varepsilon^{\kappa\pi}/\gamma^{1/3} = \beta$ are at fixed ε contained in an interval $[\kappa_-(\varepsilon), \kappa_+(\varepsilon)]$ which shrinks (logarithmically) to $1/(8\pi)$ as $\varepsilon \to 0$. It is thus contained in an interval $[\kappa_m, \kappa_M]$ of κ-values. It is enough to choose the upper limit of ε so that $\varepsilon^{\kappa_m\pi} = q_0$.

The invariant sets of $\mathcal{P}$ in the neighbourhood of $(\Lambda(q), \chi(q), \beta(q))$ may be described completely using the central manifold theorem, in the manner presented in Guckenheimer & Holmes [1983] and Marsden & McCracken [1976] . Following the instructions of these references, we may state[27]:

Lemma 8.4 *Let Σ be such that : $2\pi - \delta > |\Sigma + \pi/2| > \delta > 0$. For q small enough, there exists a neighbourhood $U \times V$ of $(\Lambda(q), \chi(q), \beta(q))$,*

$$U : \{|\Lambda - \Lambda(q)| < A\} \times \{|\chi - \chi(q)| < B\}, \qquad V : \{|\beta - \beta(q)| < C\}$$

with A,B,C independent of q such that : if $\beta < \beta(q)$, $\beta \in V$, U contains no invariant sets of $\tilde{\Pi}$; if $\beta > \beta(q)$ the invariant set consists of two points; if $\beta = \beta(q)$, the only invariant set in U is $(\Lambda(q), \chi(q))$

To see this, we introduce new coordinates

$$\xi_1 \equiv \Lambda - \Lambda(q), \qquad \xi_2 \equiv \chi - \chi(q), \qquad \xi_3 \equiv \beta - \beta(q) \tag{8.24}$$

centered at $\Xi_0 \equiv (\Lambda(q), \chi(q), \beta(q))$. The mapping $\mathcal{P}$ may be locally approximated by a Taylor expansion:

$$\begin{aligned} &\mathcal{P}_\Lambda : \xi_1 \Rightarrow \sum_{i=1}^{3} a_i\xi_i + \sum_{i\geq j=1}^{3} a_{ij}\xi_i\xi_j + ... \\ &\mathcal{P}_\chi : \xi_2 \Rightarrow \xi_3\mathcal{P}^0_\chi(\Xi_0) + b_1\xi_1 + (1 - \frac{a_2 b_1}{1 - a_1})\xi_2 + \sum_{i\geq j=1}^{3} b_{ij}\xi_i\xi_j + ... \end{aligned} \tag{8.25}$$

[27]The exercise on p.25 of Marsden & McCracken [1976] is almost the same as this Lemma

where a_1, a_2 are $O(q)$, a_3 even of $O(q^2)$ (cf.eq.(7.1a), where the dependence of β is concealed in $\tilde{\Sigma}$, cf.also (8.4c)), a_{ij} are all of $O(q)$,but nonvanishing(cf.eq.(7.1a)), $\mathcal{P}^0_\chi(\Xi_0) \approx \Pi^0(\chi_0) \neq 0$, the coefficient of ξ_2 is $1 - O(q)$ and the b_{ij} are nonvanishing, as a consequence of the condition on Σ, and of $O(1)$. For small q, the coefficient b_{22} is approximately $\partial^2\Pi^0/\partial\chi^2 \neq 0$. Let then $\xi_1\prime, \xi_2\prime$ be new coordinates linearly related to ξ_1, ξ_2 so that the linear part in ξ_1, ξ_2 of (8.25) becomes diagonal. The first eigenvalue is $O(q)$; it is relevant that $\xi_1\prime = O(1)\xi_1 + O(q)\xi_2$ so that the orders of magnitude in the transformed system:

$$\begin{aligned} \mathcal{P}_1 &: \xi_1\prime \Rightarrow c_1\xi_1\prime + a_3\prime\xi_3\prime + \sum_{i\geq j=1}^{3} a_{ij}\prime\xi_i\prime\xi_j\prime \\ \mathcal{P}_2 &: \xi_2\prime \Rightarrow \xi_3\prime(\mathcal{P}^0_\chi(\Xi_0) + O(q)) + \xi_2\prime + \sum_{i\geq j=1}^{3} b_{ij}\prime\xi_i\prime\xi_j\prime + .. \end{aligned} \tag{8.26}$$

are preserved. In (8.26) we have set $\xi_3\prime = \xi_3$. We enlarge $\mathcal{P}$ by adding to (8.26):

$$\mathcal{P}_3 : \xi_3\prime \Rightarrow \xi_3\prime. \tag{8.27}$$

The quantity $a_3\prime$ is $O(q)$, $b_{22}\prime \approx b_{22}$ and the $b_{ij}\prime$ are O(1). It is possible to find an approximation to the central manifold of the enlarged $\mathcal{P}$ (eqns.(8.26),(8.27)) (the invariant manifold tangent to the subspace of eigenvalue unity) around Ξ_0 in the form (see Guckenheimer & Holmes [1983, p.136])

$$\xi_1\prime = h_I(\xi_2\prime, \xi_3\prime) = \alpha_{22}(\xi_2\prime)^2 + \alpha_{23}\xi_2\prime\xi_3\prime + \alpha_{33}(\xi_3\prime)^2 + ... \tag{8.28}$$

The coefficients α_{ij} are obtained by equating the coefficients of like powers of $\xi_2\prime, \xi_3\prime$ in the condition:

$$\mathcal{P}_1(h_I(\xi_2\prime, \xi_3\prime), \xi_2\prime, \xi_3\prime) = h_I(\mathcal{P}_2(h_I(\xi_1\prime, \xi_2\prime, \xi_3\prime), \xi_2\prime, \xi_3\prime), \xi_3\prime) \tag{8.29}$$

It turns out that the α_{ij} are all of $O(q)$, as they are proportional to coefficients appearing in $\mathcal{P}_1$. According to the center manifold theorem (see Marsden & McCracken [1976, p.19]) all points in a (sufficiently small) neighbourhood $U \times V$ of Ξ_0 approach under iterations of $\mathcal{P}$ the center manifold, which is itself invariant. Thus the only possible invariant sets of $\mathcal{P}$ are to be found by restricting the action of $\mathcal{P}$ to it. Substituting then (eq.8.28) into the second equation (8.26) we obtain a description of the bifurcations at Ξ_0 :

$$\hat{\mathcal{P}}_2(\xi_2\prime, \xi_3\prime) : \xi_2\prime \Rightarrow \xi_3\prime(\cos(\chi(q) + \Sigma) + O(q)) + \xi_2\prime(1 + b_{23}\prime\xi_3\prime) + b_{22}\prime(\xi_2\prime)^2 + O(q) \times h.o. \tag{8.30a}$$

$$\hat{\mathcal{P}}_3(\xi_3\prime) : \xi_3\prime \Rightarrow \xi_3\prime \tag{8.30b}$$

For q small enough, eq.(8.30a) describes a saddle-node bifurcation: for $\xi_3\prime > 0$, the equation $\xi_2\prime = \hat{\mathcal{P}}_2(\xi_2\prime, \xi_3\prime)$, with $\hat{\mathcal{P}}_2$ restricted to the quadratic part has no solutions for small $\xi_3\prime$; it has one double zero for $\xi_3\prime = 0$ and two solutions for $\xi_3\prime < 0$. Thus saddle-node bifurcations of the circle map are transferred indeed to the complete mapping $\mathcal{P}$ at least for small q, i.e. for small ε.

We show that the same is true for flip bifurcations. In strict analogy to Lemma 8.3 it is true that:

Lemma 8.5 *Let $2\pi - \delta > |\Sigma - \pi/2| > \delta > 0$ and χ_0, β_0 obeying (cf.Lemma 8.3):*

$$\beta_0\Pi_0(\chi_0) = \chi_0, \qquad \beta_0\frac{\partial\Pi_0}{\chi_0}(\chi_0) = -1 \tag{8.31}$$

Then, for q small enough, the set of equations:

$$\mathcal{P}_\Lambda(\Lambda, \chi, q) = \Lambda, \qquad \mathcal{P}_\chi(\Lambda, \chi, q) = \chi, \qquad \det|\mathbb{I} + \mathbb{D}\mathcal{P}| = 0 \tag{8.32}$$

has a unique solution $(\Lambda(q), \chi(q), \beta(q))$ which tends to $(R_{L,f}, \chi_0, \beta_0)$ as $q \to 0$.

The argument is the same as in Lemma 8.3 with obvious changes of sign in (8.22) and (8.23). The same transformations of variables as in Lemma 8.4 bring $\mathcal{P}$ locally in the form (8.26) with a minus sign in front of ξ_2 in the linear part of $\mathcal{P}_2$. We inquire next whether the flip bifurcations survive the transition from one to two dimensions (from Π to $\mathcal{P}$):

Lemma 8.6 *Let Σ obey: $2\pi - \delta > |\Sigma - \pi/2| > \delta > 0$. For q small enough, there exists a neighbourhood $U \times V$ of $(\Lambda(q), \chi(q), \beta(q))$, $U : \{|\Lambda - \Lambda(q)| < A\} \times \{|\chi - \chi(q)| < B\}$, $V : \{|\beta - \beta(q)| < C\}$, with A,B,C independent of q and such that : if $\beta \le \beta(q), \beta \in V$, U contains just one invariant set of $\mathcal{P} \circ \mathcal{P}$, which is also an invariant set of $\mathcal{P}$; if $\beta > \beta(q)$ the invariant set of $\mathcal{P}$ consists of one fixed point and one stable orbit of period two.*

One may doubt *a priori* that this is the case, since pitchfork bifurcations (as present in the mapping $\Pi \circ \Pi$) are not stable under perturbations in general. Nevertheless, the special features appearing from the restriction to mappings of the form $\mathcal{P} \circ \mathcal{P}$ allows a proof of the persistence of flip bifurcations when one moves from Π to $\mathcal{P}$.

Expanding $\mathcal{P}$ around the point $(\Lambda(q), \chi(q), \beta(q))$ of Lemma 8.5 the mapping $\mathcal{P} \circ \mathcal{P}$ may be written with the notations of (8.24):

$$\begin{aligned} &\mathcal{P}_1^2 : \xi_1 \Rightarrow a_1\xi_3 + a_2\xi_1 + a_{11}\xi_1^2 + a_{22}\xi_2^2 + a_{13}\xi_1\xi_3 + a_{23}\xi_2\xi_3 + a_{33}\xi_3^2 + ... \\ &\mathcal{P}_2^2 : \xi_2 \Rightarrow \xi_2 + b_{11}\xi_1^2 + b_{12}\xi_1\xi_2 + b_{13}\xi_1\xi_3 + b_{23}\xi_2\xi_3 + b_{33}\xi_3^2 \\ &\qquad\quad + d_{222}\xi_2^3 + d_{223}\xi_2^2\xi_3 + d_{233}\xi_2\xi_3^2 + ... \\ &\mathcal{P}_3^2 : \xi_3 \Rightarrow \xi_3 \end{aligned} \tag{8.33}$$

where the a_i, a_{ij} are of $O(q)$ or less, $a_2 > 0$, $a_{12} = 0$, the b_{ij} are $O(1)$, $b_{22} = 0$, $b_{23} \neq 0$ (as $q \to 0$ it approaches $-\partial^2\Pi/\partial\beta\partial\chi = 1/\beta_0$;) and terms of $O(\xi^3)$ must be taken into account (to describe the pitchfork); the coefficient $d_{222} \neq 0$ and has a nonzero limit as $q \to 0$, as a consequence of the conditions on Σ. In (8.33) we extended $\mathcal{P}^2$ through the addition of the identity concerning ξ_3 (see Guckenheimer & Holmes [1983]). Changing variables to:

$$\xi_1 = \xi_1\prime + \frac{a_1}{1 - a_2}\xi_3, \qquad \xi_2\prime = \xi_2, \qquad \xi_3\prime = \xi_3 \tag{8.34}$$

causes the linear term $a_1\xi_3$ to disappear in the transformed equations. One looks for an invariant manifold of $\mathcal{P}^2$ tangent to the plane $\xi_1\prime = 0$ in the form (8.28) above. From an equation analogous to (8.29) one determines the coefficients α_{ij} which turn out to be of $O(q)$ or smaller. Replacing ξ_1 as a function of ξ_2, ξ_3 in the second equation (8.33) one obtains a description of the invariant sets in the approximation (8.28) of the center manifold of the extended system (8.33):

$$B_{23}\xi_2\xi_3 + B_{33}\xi_3^2 + B_{222}\xi_2^3 + \xi_3 O(\xi_2^2, \xi_2\xi_3, \xi_3^2) + O(q)(\xi_2^4 + ...) = 0 \tag{8.35}$$

where B_{23}, B_{33} are corrections of $O(q)$ to b_{23}, b_{33} of (8.33) and are of $O(1)$, B_{222} is a correction of $O(q)$ to d_{222} in (8.33), the magnitude of the coefficients of the other cubic terms is $O(1)$; it is essential that no terms with ξ_2^2 appear, even of $O(q)$: these would destroy the "'pitchfork"'; there are terms of $O(\xi_2^4)$ multiplied by $O(q)$, but these are harmless. The prime on ξ_2, ξ_3 has been dropped in (8.35), in view of (8.34). At $\xi_3 = 0$ (8.35) reduces to :

$$B_{222}\xi_2^3 + O(q)\xi_2^4 = 0 \tag{8.36}$$

which shows the triple zero at $\xi_2 = 0$ and another zero far away. A solution of (8.35) which is analytic in ξ_3 may be obtained formally by writing:

$$\xi_2 = e_1\xi_3 + e_2\xi_3^2 + ... \tag{8.37}$$

and identifying coefficients; one obtains $e_1 = -B_{33}/B_{23}$, etc. There exist also two other solutions analytic in $\sqrt{\xi_3}$ which are the continuations in ξ_3 of the other zeroes of (8.36): letting in (8.35) $\xi_3 = x^2$ and substituting there:

$$\xi_2 = f_1 x + f_2 x^2 + \ldots \tag{8.38}$$

one determines f_1 as one solution of:

$$B_{23} f_1 + B_{222} f_1^3 = 0 \tag{8.39}$$

The solution $f_1 = 0$ leads to (8.37). The coefficients B_{23},B_{222} are corrections of $O(q)$ to the derivatives $\partial^2\Pi/\partial\beta\partial\chi$, $\partial^3\Pi/\partial\chi^3$ at $(\chi_0.\beta_0)$: these are $\approx -1/\beta_0, 1$ in turn. Thus, $f_1 \approx \pm\beta_0^{-1/2}$. This shows that indeed, for $\xi_3 < 0$ there exists just one solution of (8.35) but there are three solutions for $\xi_3 > 0$, i.e. a "‘pitchfork"’. This ends the argument for Lemma 8.6.

8.3. *Conclusions*

In the statements of Lemmas 8.3 - 8.6, the values of Σ lying near the peaks of the saddle-node and flip bifurcation curves (see Fig. 11) were excluded. With this exception, we can conclude this section with

Theorem 8.1 *If $\beta > \beta_e(\Sigma)$ of Lemma 7.2, the invariant sets of the half - period Poincaré map $\mathbb{P}(\beta,\Sigma,\Lambda,\chi)$ of Duffing's equation consist, for sufficiently large Γ of fixed points and periodic points of period two only - with the possible exception of small neighbourhoods, vanishing as Γ increases, of points $(\beta = 1, \Sigma = \pm\pi/2)$. These invariant points are in one-to-one correspondence with those of the circle map $\Pi(\beta,\chi,\Sigma)$ of (7.9) and approach the latter as $\Gamma \to \infty$ $(\varepsilon \to 0)$. The bifurcation lines $\beta_F(\Sigma)$, $\beta_S(\Sigma)$ approach those of Π in this limit.*

The restriction to the domain of large β is artificial: one can extend the argument and show the stability in the transition $\Pi \Rightarrow \mathcal{P}(\equiv \mathbb{P})$ of the period doubling cascade and - presumably - of the limiting chaotic motion. This gives a natural explanation for the chaotic behaviour observed a long time ago in the damped and forced Duffing oscillators. Theorem 8.1 also gives a complete understanding of the regularities observed in the bifurcation pattern of Duffing's equation at large forcing and (moderately) high damping.

9. Comments and conclusions

It is apparent that an important ingredient in the justification of the bifurcation structure of eq.(1.1) (or eq.(1.8)) is the sudden change of "‘natural"’ reference at $t = 0$ in the description of the motion. This is brought about by the discontinuity occurring at t=0 when passing from the left hand reference $X_L(t)$ to $X_R(t)$ (see Fig.2). The continuation of $X_L(t)$ to $t > 0$ is oscillatory (see Fig.6) because (intuitively) a particle moving for t<0 at the bottom of the potential well $x^4/4 - x\sin t$ (cf.eq.(1.8))cannot follow the infinite velocity of the minimum at $t = 0$. The particle behaves as if it had been subjected to a "‘kick"’. In a series of papers by Parlitz [1993] and Parlitz *et al.* [1991a,b], the authors show that an infinite sequence of bifurcation curves occur in a very simple "‘kick and twist model"’, described by the differential equations ($x = r\cos\alpha, y = r\sin\alpha$, d>0):

$$\dot{r} = -\frac{d}{2}r \qquad \dot{\alpha} = 1 + r^2 \tag{9.1}$$

supplemented by a periodic "‘kick"’, i.e. a displacement of the y coordinate by an amount a at equally spaced time intervals T. The control parameters are the amplitude a and the period T.

For Duffing-type equations Eilenberger & Schmidt [1992] give a simple and elegant argument that for a *real* sudden change of the forcing at t=0 (obtained by replacing $\sin t$ on the right hand side of

(1.8) by a step function) a nonlinear dependence of the restoring force on the displacement $x(t)$ leads naturally at sufficiently high damping to a (half-period) Poincaré map of the form (1.4) or (7.9), i.e. to a "'circle map"'. The latter describes accurately the bifurcation structure of the Duffing equation at high forcing and damping. The critique to this argument is obviously that there is in (1.8) or (1.9) no real discontinuous change of forcing (no real "'kick"'[Eilenberger & Schmidt, 1992])at $t = 0$: the change of reference is only a convenient artefact. As is apparent from Section 6, see Fig.8 or Fig. 9, the development of the circle map occurs actually in a short time interval $\tau = O(1/\gamma)$ near $t = 0$. In the limit $\Gamma \to \infty$ this interval becomes infinitely short compared to the whole interval $[0, \pi/2]$ (actually $O(1/\ln(1/\varepsilon)$ compared to it) but is infinitely long $(O(1/(\varepsilon^{3/8} \ln 1/\varepsilon)))$ compared to the boundary layer, where τ (cf.eq.(1.18)) is $O(1)$ (this is the -somewhat enlarged - *transition* region of Schmidt & Eilenberger [1998] where the inner and outer expansions are *matched*, see Sect. 3).

The following is a qualitative argument for the appearance of the circle map as an approximation to the half-period Poincaré map - relating $t = -\pi/2$ to $t = \pi/2$ -, as it emerges from the discussion in Sects. 5 and 6. The discussion ignores the difficulties related to the changes of variable between τ and $\theta_{L,R}$ of (4.2) and assumes they can be performed all the way down to $t = 0$. In the time interval $[-\pi/2, -\tau_0\varepsilon^{3/8-s}], s > 0$ the motion around the left hand reference solution is essentially harmonic in the variable θ_L, eq.(4.2) i.e. with a period independent of the amplitude. According to Lemma 5.2 there exists an additional phase proportional to the square of the amplitude at $t = -\pi/2$ which is a cause of the distortions shown in Fig.5. To these, we have to add the displacements appearing when we get into the boundary layer down to $t = 0$ where the motion of solutions in the vicinity of $X_L(t)$ is approximately described by the linear equation (5.39).Fig.5 testifies however that the distortion of the small disk at $t = -\pi/2$ in the $t = 0$ plane is not a huge effect, so that we make the rough approximation that harmonic motion is dominant and thus all *initial* phases ψ_i - measured around $X_L(t)$ at $t = -\pi/2$ - have increased at $t = 0$ by the same amount a, independent of the amplitude:

$$\psi(0) = \psi_i + a \tag{9.2}$$

At $t > 0$ the motion consists essentially of rotations around the right hand reference solution $X_R(t)$., The angular velocity has an harmonic (amplitude-independent) term in the variable θ_R, eq.(4.2) and (to a first approximation) a second term which decreases with θ_R like $1/\theta_R^{3/4}$ and is proportional to the square of the amplitude (of the distance from the origin in the $w, dw/d\theta$ plane, see eqs.(6.15a),(refeq:6.14b). This latter term is the prominent effect of the nonlinearities in (1.1). Its effect is limited to a time $\tau < 1/\gamma$, after which the motion is essentially harmonic. The final value ψ_f of the angle at $t = \pi/2$ is the value of the rotation angle around $X_R(t)$ between $t = 0$ and $\pi/2$. Thus, for a solution rotating at distance R from $X_R(t)$ (with R measured in units of $\varepsilon^{1/8}$),

$$\psi_f = \frac{A}{\sqrt{\varepsilon}} + \frac{BR^2}{\gamma^{1/3}} + C \tag{9.3}$$

The first term is common to all solutions and is the effect of the harmonic part: $1/\sqrt{\varepsilon}$ is the order of magnitude of $t = \pi/2$ when expressed in the variable θ_R, eq.(4.2). The second term is the integral over $\theta_R^{-3/4}$ up to $\tau = 1/\gamma$; since $\theta_R \approx \tau^{4/3}$ a time $\tau = 1/\gamma$ corresponds to $\theta_R \approx 1/\gamma^{4/3}$. The third term is the contribution to the rotation angle of times larger than $1/\gamma$ and is again independent of the chosen solution.

The continuation to $t > 0$ of the left hand reference solution $X_L(t)$ also rotates around $X_R(t)$ at a distance R_L from it (cf.Fig.6) and achieves at $t = \pi/2$ a total rotation (9.3) given by an angle denoted by $\psi_{f,L}$. Neighbouring solutions with a distance $R_L + \delta R$ to $X_R(t)$ acquire a rotation angle:

$$\psi_f(\delta R) = \psi_{f,L} + \frac{2BR_L\delta R}{\gamma^{1/3}} \tag{9.4}$$

We assume R_L is constant down to $t = 0$. A solution starting at $t = -\pi/2$ close enough (i.e. $O(\varepsilon^{3/16+\kappa\pi/2})$), cf. Theorem 4.1) to $X_L(t)$ with an angle ψ_i to the x-axis ends up at $t = 0$ in a disk of radius $r = O(\varepsilon^{\kappa\pi+1/8}) = O(e^{-\Delta\pi}\varepsilon^{1/8})$ around X_L (cf.eqns.(5.52a),(5.52b)) and in a position enclosing an angle $\psi(0)$, eq.(9.2) with the x-axis. Its distance to $X_R(0)$ is:

$$R_L + \delta R = \sqrt{R_L^2 + r^2 + 2R_L r\cos(\psi + a)} \approx R_L + r\cos(\psi_i + a) \tag{9.5}$$

which gives an estimate of δ_R. Substituting in (9.4) one obtains the circle map, which gives the angle ψ_f at $t = \pi/2$ in terms of ψ_i at $t = -\pi/2$:

$$\psi_f = \psi_{f,L} + \frac{const \times \varepsilon^{\kappa\pi}}{\gamma^{1/3}}\cos(\psi_i + a) \tag{9.6}$$

This mapping is equivalent to the form (1.4) or (7.9) (cf.Sect.7.1).The whole paper is actually devoted to the justification of this qualitative picture in a correct manner.

The role of the nonlinearities may be appreciated if one compares eq.(1.8) with a (possible) linear version of it:

$$\varepsilon\ddot{x} + 2\mu\dot{x} + x = (\sin t)^{1/3} \tag{9.7}$$

which has, for all ε,μ a unique periodic solution. One can perform for it the same analysis with inner and outer expansion as for (1.8)[28]. The difference $u(t) \equiv x(t) - x_L(t)$ to a corresponding reference solution $x_L(t)$ obeys the equation of a linear damped harmonic oscillator;the latter transforms a small disk $u^2 + \dot{u}^2/\varepsilon < r^2$ at $t = -\pi/2$ into a disk of radius smaller by a factor $\varepsilon^{\kappa\pi}$ at $t = \pi/2$ (around a corresponding reference $x_R(t)$), almost without change of shape (to first order in $\mu/\sqrt{\varepsilon}$). The difference in rotation angles for different amplitudes, as expressed by (9.4) in the nonlinear case, is zero.

The author believes it is a special virtue of the averaging method of Bogolyubov & Mitropolski [1961] that it allows a systematic and easily interpretable treatment of the nonharmonic behaviour in the small $|t|$ domain (the "‘transition region"’ of Schmidt & Eilenberger [1998]). In fact, use of this method makes up the main difference between the treatment of my earlier internal report[1990] and of the present work to that of the papers of G.Eilenberger and K.Schmidt[1992],[1998]. The analysis of these authors is based on the adiabatic theorem of classical mechanics [Arnold, 1978], [Landau & Lifshitz, 1960], applied to the motion described by eq.(1.9) around the reference ("‘creeping"’) solutions $X_L(t\sqrt{\varepsilon}), X_R(t\sqrt{\varepsilon})$[29].It is not so easy to extend the adiabatic approximation to the region of small $|t|$ in such a manner that it matches there to the boundary layer description of the motion, given to zeroth order by eq.(3.24a). This is done by a method of time-dependent canonical transformations, specially devised fot this purpose and presented in the Appendix B of Schmidt & Eilenberger [1998]. The procedure is claimed to be numerically successful but it is difficult to identify in it the various terms given by the averaging method of Sects. 5.1 and 6.

Both the work of G.Eilenberger and K.Schmidt[1992; 1998] and the present one (with its earlier version) rely on a Taylor expansion to first (or second order) of the Poincaré map around the continuation of the left hand reference solution to $t > 0$.This statement may not be obviously valid for the work of Eilenberger & Schmidt because of the different formulation of the Poincaré map, but a moment's consideration shows that it is implied in eqns.(5.8),(5.9) of Eilenberger & Schmidt [1992] and in eq.(24) of Schmidt & Eilenberger [1998] . The reason why a Taylor expansion is expected to be sufficient is that the magnitude of the small disk of radius $\varepsilon^{\kappa\pi/2+3/16}$ is further reduced by a factor[30] $\varepsilon^{\kappa\pi/2}$ at $t = 0$, so that one is interested in a "‘really"’ small neighbourhood of the continuation of $X_L(t)$ at $t > 0$.

For given Δ and Γ in the large $\Delta - \Gamma$ region considered here (or ε and μ in the corresponding domain) one determines numerically the coefficients of this expansion (as in Schmidt & Eilenberger [1998]) or

[28] inner variables X, τ may be defined through $x \equiv \varepsilon^{1/6}X, t = \varepsilon^{1/2}\tau$

[29] the latter are introduced in a manner similar to this paper and to the work of Byatt-Smith [1987]

[30] the factor $\varepsilon^{3/16}$ disappears through rescaling and Liouville transformation

integrates the variational equation around $X_L(t)$ from $t = 0$ to $t = \pi/2$, as analyzed (in principle) in this work (and in the earlier report);the advantage of the latter method is that the averaging transformations allow several statements about the solutions (see Sect.6.2), especially about their behaviour as $\varepsilon \to 0$. Really "‘universal"’ numerical constants appear only in this limit (cf.eqns.(6.45),(6.46)). They are determined by the boundary layer equations in the limit $\gamma \to 0$.This paper devotes much attention to the justification of these limits (see Lemma 6.4), the reason being that not all of them *exist*: see eqns.(6.55),(6.56). It is in fact the equilibrium between this divergence and the magnitude of the small disk at $t = 0$ (see Fig. 5) which is responsible for the onset of bifurcations as the damping decreases at fixed Γ. Unfortunately a similar discussion appears to be absent in the work of Eilenberger & Schmidt.

It is remarkable that the circle map - which comes from the *first* order Taylor expansion - turns out, according to the numerical evidence of Schmidt & Eilenberger [1998], to have a large domain of validity - at least qualitatively - in the $\Gamma - \Delta$ plane, not limited to values of the damping increasing logarithmically with the forcing (as assumed in this paper). It is an open question to what extent (down to which value of the damping Δ) the very rich bifurcation structure of the circle map is indeed transferred to the highly complex bifurcation diagram of the Duffing equation in the $\Gamma - \Delta$ plane. The arguments of Sect.8.1 only show that the transfer of the upper part of the bifurcation curves does occur asymptotically in Γ.

Acknowledgements

This paper - which is a new formulation of my internal report Stefanescu [1990] - originated in discussions with G.Höhler more than twenty years ago. He was engaged at that time on a numerical study of the bifurcation diagram of the Duffing equation (Ref.Höhler [1993]). A (very) short exchange of E-Mails with J.Gallas last year determined me to reconsider my work of that time and look at it carefully again. I wish to thank him for this correspondence.

All calculations (numeric and symbolic) were performed with a MAPLE10 program.

A. Appendix A: A Bound on the Increase of $E(\theta)$ in the Interval $(\varepsilon^{3p}, \varepsilon^q)$

In this Appendix we drop the index R on θ_R, w_R, θ_{0R}, g_R because the variables for the interval $(-\pi/2, 0)$ do not occur at all. The change of variables $W = wk(\theta)$ transforms equation (4.13) into:

$$\frac{d^2W}{d\theta^2} - \frac{2}{k^2}\frac{dk}{d\theta}\frac{dW}{d\theta} + W(1 + G(\theta)) + W^2 + \frac{W^3}{3} = 0 \tag{A.1}$$

where

$$G(\theta) \equiv g(\theta) + \frac{2}{k^2}\left(\frac{dk}{d\theta}\right)^2 - \frac{1}{k}\frac{d^2k}{d\theta^2} \tag{A.2}$$

This function decreases like $1/\theta^2$ for large θ, independently of the choice of p, eq.(4.11) in paragraph 4.5. The energy associated to (A.1):

$$\mathcal{E}(\theta) = \frac{1}{2}\left(\frac{dW}{d\theta}\right)^2 + \frac{W^2}{2}(1 + G(\theta)) + \frac{W^3}{3} + \frac{W^4}{12} \tag{A.3}$$

evolves in time according to:

$$\frac{d\mathcal{E}}{d\theta} = \frac{W^2}{2}\frac{dG}{d\theta} - \frac{3}{4\theta}\left(\frac{dW}{d\theta}\right)^2 \tag{A.4}$$

where we have used the approximation $k(\theta) \approx \varepsilon^{3p/2}/t^{1/2}$ for small t. The energy $\mathcal{E}$ is bounded because, using $W^2 < 6\mathcal{E}$ (cf.eq.(4.16)), eq.(A.4) leads to the inequality:

$$\frac{d \ln \mathcal{E}}{d\theta} < const \left| \frac{dG}{d\theta} \right| \tag{A.5}$$

which means $\mathcal{E} < const \mathcal{E}(\theta_0)$[31].This is still a very weak bound for $w(\theta)$: it simply implies that $|w(\theta)| < const/k(\theta)$ i.e. using (4.23)

$$|w(\theta)| < const \sqrt{\mathcal{E}(\theta)} \left(\frac{\theta}{\theta_0}\right)^{3/8} < const \sqrt{\mathcal{E}(\theta_0)} (\theta/\theta_0)^{3/8}. \tag{A.6}$$

We consider now in more detail the negative "'damping "' term in (A.4): by comparing with the motion $W_f(\theta)$ in the time independent potential:

$$V(W) = \frac{W^2}{2} + \frac{W^3}{3} + \frac{W^4}{12} \tag{A.7}$$

we shall show that, in fact, the energy $\mathcal{E}$ decreases to zero like $const/\theta^s$, for some $s > 0$. Let $T(\mathcal{E})$ be the period of the motion with energy $\mathcal{E}$ in the potential $V(W)$, eq.(A.7) and let $T_M \equiv \sup T(\mathcal{E})$ over all $\mathcal{E} > 0$ [32].Let further T be a time interval obeying $T > qT_M$ for some integer q>1. The following is useful:

Statement A.1 *There exists a constant k_T, $0 < k_T < 1$ so that the inequality:*

$$\left(\frac{dW}{d\theta}\right)^2 > \mathcal{E} \tag{A.8}$$

is fulfilled by the motion $W_f(\theta)$ with energy $\mathcal{E}$ in the fixed potential $V(W)$, eq.(A.7) during a time $k_T T$, independently of the energy of the motion.

Indeed, we evaluate first the fraction $\tilde{k}(\mathcal{E})$ of a period of the motion with energy $\mathcal{E}$ during which (A.8) is obeyed. Let first $\mathcal{E} > 1$:

$$\tilde{k}(\mathcal{E}) = \frac{I(\mathcal{E}/2)}{I(\mathcal{E})} \tag{A.9}$$

with

$$I(f\mathcal{E}) \equiv \int_{W(-,f\mathcal{E})}^{W(+,f\mathcal{E})} \frac{dW}{\sqrt{\mathcal{E} - V(W)}} = \frac{1}{\mathcal{E}^{1/4}} \int_{u(-,\lambda)}^{u(+,\lambda)} \frac{du}{\sqrt{1 - \lambda^2 u^2/2 - \lambda u^3/3 - u^4/12}}$$

where $W(-/+, \mathcal{E})$ are the two real roots of the equation $V(W) = f\mathcal{E}$, $u(-/+, \lambda)$ are the roots of the corresponding equation after changing variables to $W = u\mathcal{E}^{1/4}$, $f = 1/2, 1$ and $\lambda = 1/\mathcal{E}^{1/4}$. The function $\tilde{k}(\mathcal{E}(\lambda))$ is a continuous, strictly positive function of λ on the closed interval $[0, 1]$ and achieves there its smallest value which is different from zero (and less than 1).The reasoning may be repeated for $\mathcal{E} < 1$, with the change of variables $W = u\mathcal{E}^{1/2}$. Let the minimum of $\tilde{k}(\mathcal{E})$ over the whole range $\mathcal{E} > 0$ be denoted by $\tilde{k}_m$. The θ-time interval T contains $r \geq q$ complete periods of the motion with energy $\mathcal{E}$, so that the time interval in which (A.8) is obeyed is at least $r\tilde{k}_m T/(r + 1)$. Then, choosing:

$$k_T = \inf_{r \geq q} \frac{r\tilde{k}_m}{r + 1} = \frac{q}{q + 1} \tilde{k}_m$$

statement A.1 is verified.

[31] cf. eqn.(4.23); in this section we drop the index R

[32] $T(\mathcal{E} \to 0$ as $\mathcal{E} \to \infty$

Now, consider the two equations (A.1) and:

$$\frac{d^2W_f}{d\theta^2} + W_f + W_f^2 + \frac{W_f^3}{3} = 0 \tag{A.10}$$

and motions $W(\theta)$, obeying (A.1), and $W_f(\theta)$, obeying (A.10), such that they have the same initial conditions at some "‘initial"’ θ-time $\theta_i > \theta_0$. If we subtract (A.10) from (A.1), use the notations:

$$\rho = W_f - W, \qquad \sigma = \frac{dW_f}{d\theta} - \frac{dW}{d\theta}, \qquad \eta = |\rho| + |\sigma|, \tag{A.11}$$

and integrate the first order differential equations equivalent to (A.1) and (A.10) from θ_i to θ, we obtain the inequality [33]:

$$\begin{aligned} \eta \leq \int_{\theta_i}^{\theta} \eta d\theta + \int_{\theta_i}^{\theta} \eta |W + W_f| d\theta + \int_{\theta_i}^{\theta} \frac{\eta}{3} |W^2 + WW_f + W_f^2| d\theta \\ + \int_{\theta_i}^{\theta} \frac{3}{4\theta} |\frac{dW}{d\theta}| d\theta + \int_{\theta_i}^{\theta} |WG(\theta)| d\theta \end{aligned} \tag{A.12}$$

Because the energy $\mathcal{E}$ of the "‘true"’ motion $W(\theta)$ is bounded, all terms containing W, W_f explicitly are bounded by constants, so that we can represent (A.12) by:

$$\eta < M \int_{\theta_i}^{\theta} \eta d\theta + M_1 \int_{\theta_i}^{\theta} (\frac{1}{\theta} + G(\theta)) d\theta \equiv M \int_{\theta_i}^{\theta} \eta d\theta + M_1 H(\theta) \tag{A.13}$$

with M, M_1 suitable constants and $H(\theta)$ monotonically increasing and positive, $H(\theta_i) = 0$. For $\theta_i < \theta < \theta_i + T$, $H(\theta) \leq const \times T/\theta_i$. Gronwall's inequality [Bellman, 1953] p.35, [Coddington & Levinson, 1955],p.37 implies then, for $\theta_i < \theta < \theta_i + T$:

$$\eta(\theta) < M_1 H(\theta) \exp(MT) < const \frac{T}{\theta_i} \tag{A.14}$$

where the constant is independent of θ_i. We estimate now the energy *loss* $\Delta\mathcal{E}$ of the "‘true"’ motion $W(\theta)$ in a "‘$\theta - time$"’T:

$$\begin{aligned} \Delta\mathcal{E} = \int_{\theta_i}^{\theta_i+T} \frac{3}{4\theta} \left(\frac{dW}{d\theta}\right)^2 d\theta = \int_{\theta_i}^{\theta_i+T} \frac{3}{4\theta} \left(\frac{dW_f}{d\theta}\right)^2 d\theta + \\ \int_{\theta_i}^{\theta_i+T} \frac{3}{4/\theta} \left(\left(\frac{dW}{d\theta}\right)^2 - \left(\frac{dW_f}{d\theta}\right)^2\right) > (k_T T) \frac{3}{4\theta_i} \mathcal{E}(\theta_i) \\ - C\sqrt{\mathcal{E}} \int_{\theta_i}^{\theta_i+T} \eta(\theta) \frac{3}{4\theta} d\theta > k_T T \frac{3}{4(\theta_i + T)} \mathcal{E} - C_2 \frac{T^2}{\theta_i^2} \sqrt{\mathcal{E}} \end{aligned} \tag{A.15}$$

In the first step we used statement A.1 and inequalities like (4.16) to bound $|dW/d\theta|$, $|dW_f/d\theta|$ from above. Eqn.(A.14) was used in the second step. The total change of energy $\mathcal{E}(\theta_i + T) - \mathcal{E}(\theta_i)$ is obtained by adding the increase due to the first term in (A.2). This latter is bounded by $C_1 T \times \mathcal{E}(\theta_i)/\theta_i^3$. One verifies that, if θ_i is sufficiently large, the total change of energy is negative. We may even require that it be larger in absolute value than $(3/4) k_1 \mathcal{E} T/\theta_i$, for a number $0 < k_1 < k_T$. This gives a lower bound $\mathcal{B}$ on the energies for which this may occur ($\theta_i/(\theta_i + T) \approx 1$):

$$\sqrt{\mathcal{E}} > \sqrt{(}\mathcal{B}) \equiv \frac{4}{3} \frac{C_2 T}{\theta_i} \frac{1}{k_T - k_1 - 4C_1/3\theta_i^2} \tag{A.16}$$

[33] $\eta = 0$ at θ_i

This bound depends on θ_i and decreases like $1/\theta_i^2$.We may assume that the maximal energy $\mathcal{E}_m$ at θ_i is such that (A.16) is satisfied and that even, say,[34]:

$$\mathcal{E}_m(\theta_i) > 2\mathcal{B} \tag{A.17}$$

Then at $\theta = \theta_i + T$ the energies $\mathcal{E}$ of all motions for which (A.16) is true at θ_i have decreased at least to $\mathcal{E}(1-(3/4)k_1T/\theta_i)$, and so has the maximal

$$\mathcal{E}_m(\theta_i + T) = \mathcal{E}_m(\theta_i)(1 - \frac{3}{4}\frac{k_1 T}{\theta_i}) \tag{A.18}$$

This is a bound for the energies of all motions with $\mathcal{E}(\theta_i) < \mathcal{E}_m(\theta_i)$: indeed, all those motions for which the inequality (A.16) at θ_i is not obeyed cannot acquire by (A.5) in the θ-time T sufficient energy to get over $\mathcal{E}_m(\theta_i + T)$, in view of the condition (A.17) if θ_i is large enough. Further, the decrease of the maximal energy $\mathcal{E}_m(\theta)$ in the interval $(\theta_i, \theta_i + T)$ is less than that of the bound $\mathcal{B}$ in the same θ-interval: at $\theta_i + T$ the latter is, according to (A.16)

$$\mathcal{B}(\theta_i + T) \approx \mathcal{B}(\theta_i)(1 - 2T/\theta_i) \tag{A.19}$$

Thus, (A.17) is obeyed also at $\theta_i + T$ with the maximal energy (A.18) and the bound $\mathcal{B}$ of (A.19);we may then proceed to $\theta_i + 2T$, etc. and conclude that, after n steps, the maximal energy is bounded by:

$$\mathcal{E}_m(\theta_i + nT) = \mathcal{E}_m(\theta_i)\prod_{j=1}^{n}(1 - \frac{3}{4}\frac{k_1 T}{\theta_i + jT}) \tag{A.20}$$

For large n, the product in (A.20) behaves like $(nT/\theta_i)^{-3k_1/4} \approx (\theta/\theta_i)^{-3k_1/4}$. We conclude that the maximal energy $\mathcal{E}_m$ decreases like $(\theta/\theta_i)^{-r}$, with $0 < r < 1$. As a consequence, the weak bound on $|w(\theta)|$ contained in the first inequality of (A.6) may be now strengthened to:

$$|w(\theta)| < const\left(\frac{\theta}{\theta_i}\right)^{3/8 - r/2} \tag{A.21}$$

We return now to the energy $E(\theta)$ of the original equation (4.13) and to the inequality (4.25):

$$\frac{dE}{d\theta} < \frac{1}{3}\left|w(\theta)^3 \frac{dk}{d\theta}\right| < const \times \left(\frac{\theta}{\theta_i}\right)^{9/8 - 3r/2}\left(\frac{\theta_i}{\theta}\right)^{3/8}\frac{1}{\theta} \tag{A.22}$$

Integration of (A.22) leads to:

$$E(\theta) - E(\theta_i) < const \times \frac{1}{3/4 - 3r/2}\left(\frac{\theta}{\theta_i}\right)^{3/4 - 3r/2} \tag{A.23}$$

which justifies our assertion in eq.(4.27)(with $s = 3r/2$).

B. The Inversion of the Averaging Transformations

A simple proof is offered that the averaging transformations (5.7a), (5.7b) (or (5.11a), (5.11b)) leading from (R, ϕ) to (R_1, ϕ_1) (or from the latter to (R_2, ϕ_2)) are invertible, if the quantity $h(\theta_L)$, eqns.(4.9), (5.3) is small enough. This is achieved either for small enough ε or large enough $|\theta_L|$. In Section 6 the quantity $h(\theta_L)$ is replaced in the averaging transformations by $k(\theta_R)$, which is monotonically

[34] it is at our disposal to increase $\mathcal{E}(\theta_i)$, if necessary

decreasing like $\theta_R^{-3/8}$ and may be thus made small for large θ_R. We show that the transformation (5.7a),(5.7b) is one-to-one from a strip $\{0 \leq R \leq M\}$ in the (R, ϕ) plane to its domain of values in (R_1, ϕ_1), for small h (or k). Here M is the bound on the values of R established in Section 4.7(see Corollary 4.1).

From (5.7a),(5.7b), taking derivatives at fixed θ_L one verifies that, for (small) positive constants $C_{ij}, i, j = 1, 2$, it is true that:

$$|\frac{\partial R_1}{\partial R}| \geq 1 - C_{11}h(\theta_L)M, \qquad |\frac{\partial R_1}{\partial \phi}| \leq C_{12}h(\theta_L)M^2 \tag{B.1a}$$

$$|\frac{\partial \phi_1}{\partial R}| \leq C_{21}h(\theta_L), \qquad |\frac{\partial \phi_1}{\partial \phi}| \geq 1 - C_{22}h(\theta_L)M \tag{B.1b}$$

Assume now two different points (R_a, ϕ_a), (R_b, ϕ_b) of the (R, ϕ) plane were mapped to the same (R_1, ϕ_1). Consider then the two functions of s, 0 < s <1:

$$\tilde{R}_1(s) \equiv R_1(R_a + sD\cos\alpha, \phi_a + sD\sin\alpha), \qquad \tilde{\phi}_1(t) \equiv \phi_1(R_a + sD\cos\alpha, \phi_a + sD\sin\alpha) \tag{B.2}$$

where D is the distance between the two points and $\tan\alpha$ is the slope of the line joining them. Since the two functions $\tilde{R}_1(s)$, $\tilde{\phi}_1(s)$ assume the same value at $s = 0$ and $s = 1$, there exist values $0 < s_R, s_\phi < 1$ so that their derivatives with respect to s vanish there. Suppose α is such that, e.g. $|\cos\alpha| \geq 1/\sqrt{2}$. Then, using (B.1a):

$$\begin{aligned} |\frac{d\tilde{R}_1}{ds}(s_R)| >& D|\frac{\partial R_1}{\partial R}\cos\alpha| - D|\frac{\partial R_1}{\partial \phi}\sin\alpha| \\ >& D(\frac{1}{\sqrt{2}} - C_{11}hM - C_{12}hM^2) \end{aligned} \tag{B.3}$$

It is clear that for h small enough, the right hand side of (B.3) does not vanish, which contradicts the fact that $\tilde{R}_1(s)$ assumes the same value at $s = 0$ and 1.

If $|\cos\alpha| < 1/\sqrt{2}$, we use the function $\tilde{\phi}_1(s)$ of (B.2) and relation (B.1b) and reach the same conclusion.

In calculations, the inversion is achieved by expanding $R(R_1, \phi_1), \phi(R_1, \phi_1)$ in powers of h (cf.eqns. (5.9a), (5.9b)).

C. The Solutions of the Variational Equation and the WKB Approximation

The following is an adaptation to the present situation of procedures that are common in the discussion of the WKB method (see, e.g.Langer [1949] and any classical book on differential equations, e.g.Coddington & Levinson [1955])

C.1. *The existence of some special solutions with WKB asymptotics*

Consider the WKB functions (5.40):

$$V_{c,s}^{(as)}(\tau, \varepsilon) = \frac{3^{1/4}}{\Xi(\tau, \varepsilon)^{1/4}} \{\cos / \sin\} (\int_\tau^{\tau_a} \Xi(\tau, \varepsilon)^{1/2} d\tau) \tag{C.1}$$

for some $\tau_a < 0$. The functions $w_{c,s}(\theta, \tau_a) \equiv \Xi^{1/4} V_{c,s}^{(as)}$ are the solutions $\cos(\theta - \theta_a)$, $\sin(\theta - \theta_a)$ of the equation

$$\frac{d^2 w}{d\theta^2} + w = 0, \qquad \theta = -\int_\tau^{\tau_0} \Xi^{1/2}(\tau\prime) d\tau\prime \tag{C.2}$$

and[35] $\theta_a \equiv \theta(\tau_a)$. The same changes of dependent and independent variable transform the variational equation (5.39) into

$$\frac{d^2\tilde{w}}{d(\theta)^2} + \tilde{w}(1 + \tilde{G}(\theta)) = 0, \qquad \tilde{w} \equiv \Xi^{1/4} V \tag{C.3}$$

with[36]

$$\tilde{G} = \frac{5}{16}\frac{(d\Xi/d\tau)^2}{\Xi^3} - \frac{d^2\Xi/d\tau^2}{\Xi^2} \tag{C.4}$$

Solutions of (C.3) which assume at $\tau = -\varepsilon^{-\delta}$ the values and derivatives of $w_{c,s}$ are obtained using the method of variation of paramaters as the unique solutions of the linear integral equation:($\theta(\varepsilon^{-\delta}) \approx -\varepsilon^{-4\delta/3}$

$$\tilde{w}_{c,s} = w_{c,s} + \int_{-\varepsilon^{-4\delta/3}}^{\theta} \tilde{G}(\theta\prime) \sin(\theta - \theta\prime) \tilde{w}_{c,s}(\theta\prime) d\theta\prime \tag{C.5}$$

Since $\tilde{G}(\theta) \approx 1/\theta^2$, the desired solution of (C.5) may be obtained by iteration at large $|\theta|$ starting from $w_{c,s}$; since its values and derivative are bounded for $|\theta|$ sufficiently large (C.5), implies that, for such θ,

$$|(\tilde{w}_{c,s} - w_{c,s})(\theta)|, \left|\frac{d\tilde{w}_{c,s}}{d\theta} - \frac{dw_{c,s}}{d\theta}\right| < \frac{C}{|\theta|} \tag{C.6}$$

Reverting to the variable τ, to the original $V_{c,s}^{(as)}$ of (C.1) and to the sought solutions $V_{c,s}(\tau, \varepsilon)$ of (5.39):

$$V_{c,s}(\tau(\theta), \varepsilon) \equiv \frac{\tilde{w}_{c,s}(\theta)}{\Xi(\tau(\theta), \varepsilon)^{1/4}} \tag{C.7}$$

we may state that, for all $|\tau|$ sufficiently large (i.e. even larger than $\varepsilon^{-\delta}$):

$$|(V_{c,s} - V_{c,s}^{(as)})(\tau)| < \frac{C}{|\tau|^{3/2}}, \qquad \left|\left(\frac{dV_{c,s}}{d\tau} - \frac{dV_{c,s}^{(as)}}{d\tau}\right)(\tau)\right| < \frac{C}{\tau^{7/6}} \tag{C.8}$$

where we have used $d\theta/d\tau \approx \tau^{1/3}$. Eq.(C.8) shows in what sense $V_{c,s}(\tau)$ *asymptotically* approach $V^{(as)}(\tau)$ The solutions $V_{c,s}$ obtained by (C.5) for large $|\tau|$ may be extended down to $\tau = 0$.

C.2. *The limit* $\varepsilon \to 0$

The solutions $V_{c,s}$ obtained above depend on the chosen value of ε. It is, however, plausible, as shown in paragraph 5.3 (cf.eq.(5.42)) that they approach a limit as $\varepsilon \to 0$. Indeed, on an interval $[-\varepsilon^{-\delta}, 0]$ with $\delta = 3r/8$ (cf.eq.(5.22)) the difference $|\eta_L(\tau) - \eta_{00L}(\tau)|$ tends to zero as $\varepsilon \to 0$. The function $\tilde{G}$, eq.(C.4) depends on ε through $\eta_L(\tau)$ (cf.5.41), is well defined for $\varepsilon = 0$ (replacing η_L by η_{00L}) and eqn.(C.5) has a formal limit for $\varepsilon = 0$:

$$\tilde{w}_{c,s}(\theta, 0) = w_{c,s}(\theta) + \int_{-\infty}^{\theta} \tilde{G}(\theta\prime, 0) \sin(\theta - \theta\prime) \tilde{w}_{c,s}(\theta\prime, 0) d\theta\prime \tag{C.9}$$

The solution $\tilde{w}_{c,s}(\theta, 0)$ of (C.9) may be obtained by iteration and is bounded for large θ; we may estimate *at a fixed value of* θ its departure from the solution $\tilde{w}_{c,s}(\theta, \varepsilon)$ of (C.5) by Gronwall's Lemma as

[35]The definition of θ in (C.2) differs from the one of θ_L in (4.2) by the implicit consideration of the term γ^2 in the definition of Ξ in (5.41)

[36]the function $\tilde{G}$ is equal to the first two terms in (4.10) if one uses the approximation (5.41) for $\Xi(\tau, \varepsilon)$

follows: we subtract (C.9) from (C.5), separate out the τ-interval $(-\infty, -\varepsilon^{-\delta})$ and estimate $\Delta\tilde{w}_{c,s}(\theta) \equiv \tilde{w}_{c,s}(\theta,\varepsilon) - \tilde{w}_{c,s}(\theta,0)$:

$$|\Delta\tilde{w}_{c,s}(\theta)| < const \times \int_{-\infty}^{\varepsilon^{-4\delta/3}} \tilde{G}(\theta\prime,0)d\theta\prime + const \times \int_{\varepsilon^{-4\delta/3}}^{\theta} |\tilde{G}(\theta\prime,0) - \tilde{G}(\theta\prime,\varepsilon)|d\theta\prime$$
$$+ \int_{-\varepsilon^{-4\delta/3}}^{\theta} \tilde{G}(\theta\prime,\varepsilon)|\Delta\tilde{w}_{c,s}(\theta\prime)|d\theta\prime \tag{C.10}$$

Since $\tilde{G} < C/\theta^2$, the first term is bounded by $const \times \varepsilon^{4\delta/3}$. Using (C.4) and the expansions (3.9),(3.11),(cf.eq.(3.34)) one verifies that:

$$|\tilde{G}(\theta,\varepsilon) - \tilde{G}(\theta,0)| = O(\frac{\gamma}{\tau^{13/3}}, \frac{\varepsilon^{3/4}}{\tau^{2/3}})$$

so that eqn.(C.10) may be rewritten in short:

$$|\Delta\tilde{w}_{c,s}(\theta)| < \tilde{C}(\varepsilon) + C\int_{-\varepsilon^{-4\delta/3}}^{\theta} \frac{1}{\theta\prime^2}|\Delta\tilde{w}(\theta\prime)|d\theta\prime \tag{C.11}$$

where $\tilde{C}(\varepsilon)$ tends to zero when $\varepsilon \to 0$. Gronwall's Lemma[Bellman, 1953],[Coddington & Levinson, 1955],[Guckenheimer & Holmes, 1983],ch.IV shows that

$$|\Delta\tilde{w}(\theta)| < \tilde{C}(\varepsilon)\exp(\int_{-\varepsilon^{-4\delta/3}}^{\theta} \frac{C}{\theta\prime^2}) < C_1(\varepsilon) \tag{C.12}$$

where $C_1(\varepsilon)$ vanishes as ε tends to zero. This shows that, as announced, if the value of θ is kept unchanged, $w_{c,s}(\theta,\varepsilon)$ approach $w_{c,s}(\theta,\varepsilon=0)$ as $\varepsilon \to 0$. The same is true for the derivatives $dw_{c,s}/d\theta(\theta,\varepsilon)$. However, this does not yet imply that this limit exists *at fixed* τ: indeed the relation (C.2) defining θ in terms of τ depends through Ξ , eqn.(5.41), on ε, and if τ is fixed, $\theta(\tau,\varepsilon)$ is different from $\theta(\tau,\varepsilon=0)$. We estimate the difference $\Delta\theta(\tau,\varepsilon)$ between the θ-values corresponding to the same τ at ε and at $\varepsilon = 0$:

$$\Delta\theta \equiv \theta(\tau,\varepsilon) - \theta(\tau,0) = \int_{\tau_0}^{\tau} (\Xi(\tau\prime,\varepsilon) - \Xi(\tau\prime,0))d\tau\prime = O\left(\frac{\gamma}{\tau^{1/3}}, \varepsilon^{3/4}\tau^{10/3}\right) \tag{C.13}$$

where we have used the expansions (3.9),(3.11) to evaluate the integral. This difference vanishes as $\varepsilon \to 0$ for $|\tau| < \varepsilon^{-\delta}$ with the choice of δ in (C.13). We can now write, using $\theta_\varepsilon \equiv \theta(\tau,\varepsilon)$, $\hat{\theta} \equiv \theta(\tau,0)$

$$|\tilde{w}_{c,s}(\tau,\varepsilon) - \tilde{w}_{c,s}(\tau,0)| \le |\tilde{w}_{c,s}(\theta_\varepsilon,\varepsilon) - \tilde{w}_{c,s}(\theta_\varepsilon,0)| + |\tilde{w}_{c,s}(\theta_\varepsilon,0) - \tilde{w}_{c,s}(\hat{\theta},0)| \equiv T_1 + T_2 \tag{C.14}$$

The first term in (C.14) is the difference *at fixed* θ and vanishes as $\varepsilon \to 0$, according to (C.12). For the second term we use the estimate (C.14) of $\Delta\theta$ and the integral equation (C.9):

$$T_2 \le |w_{c,s}(\theta_\varepsilon) - w_{c,s}(\hat{\theta})| + \int_{-\infty}^{\theta_\varepsilon} \tilde{G}(\theta\prime)(\sin(\theta_\varepsilon - \theta\prime) - \sin(\hat{\theta} - \theta\prime)\tilde{w}_c, s(\theta\prime,0)d\theta\prime$$
$$+ \int_{\hat{\theta}}^{\theta_\varepsilon} \tilde{G}(\theta\prime)\sin(\hat{\theta} - \theta\prime)\tilde{w}_c, s(\theta\prime,0)d\theta\prime \tag{C.15}$$

All terms in (C.15) may be majorized by $const \times |\theta_\varepsilon - \hat{\theta}|$ so that, as announced, at any fixed τ in $[\varepsilon^{-\delta}, 0]$, the solutions of (C.5) approach those of (C.9) as $\varepsilon \to 0$.

C.3. *The difference of two solutions of the variational equation*

We need sometimes an estimate of the difference of two solutions $V_i(\tau)$,$(i = 1, 2)$ of the variational equation on the whole interval $[-\varepsilon^{-\delta}, 0]$, knowing the difference of their initial values (α_i, β_i), $i = 1, 2$ at $\tau = -\varepsilon^{-\delta}$. Let $\tilde{w}_i(\theta) \equiv V_i \Xi^{1/4}$ be then the solutions corresponding to them of two equations like (C.5),containing suitable initial conditions at $\tau = -\varepsilon^{-\delta}$. These latter, which we call $\hat{\alpha} \equiv w(-\varepsilon^{-\delta})$, $\hat{\beta} \equiv dw/d\theta(-\varepsilon^{-\delta})$ are obtained from (α_i, β_i) through:

$$\hat{\alpha}_i = 3^{1/4}\alpha_i\varepsilon^{-\delta/6}, \qquad \hat{\beta}_i = 3^{-1/4}\varepsilon^{\delta/6}\beta_i + \frac{3^{-5/4}}{2}\varepsilon^{7\delta/6}\alpha_i \tag{C.16}$$

Subtracting the two equations (C.5) corresponding to $\tilde{w}_i$, $i = 1, 2$ one obtains with an obvious notation:

$$\Delta\tilde{w}(\theta) = \Delta\hat{\alpha}w_c(\theta) + \Delta\hat{\beta}w_s(\theta) + \int_{-\varepsilon^{-4\delta/3}}^{\theta} \tilde{G}(\theta\prime)\sin(\theta - \theta\prime)\Delta\tilde{w}(\theta\prime)d\theta\prime \tag{C.17}$$

In a well known manner, from (C.17) and the boundedness of w_c, w_s, Gronwall's Lemma [Bellman, 1953],[Coddington & Levinson, 1955] implies that:

$$|\Delta\tilde{w}(\theta)|, \left|\Delta\frac{d\tilde{w}}{d\theta}(\theta)\right| = O(\max(\Delta\hat{\alpha}, \Delta\hat{\beta})) \tag{C.18}$$

The estimate (C.18) holds as long as $\Xi \neq 0$, in our case down to $\tau = 0$. Reverting to the original $V_i(\tau)$ and noticing that for finite τ, $\Xi(\tau)$ is finite, together with its derivative, we conclude that the same estimate (C.18) is true also for the differences of $V_i, dV_i/d\tau$. Using (C.16):

$$|\Delta V(\tau)|, \left|\Delta\frac{dV}{d\tau}(\tau)\right| = O(\max(\Delta\alpha\varepsilon^{-\delta/6}, \Delta\beta\varepsilon^{\delta/6})) \tag{C.19}$$

D. The Limit of $R_{L,f}(\varepsilon)$ for $\varepsilon \to 0$

We show that the values $R_{L,f}(\varepsilon) \equiv R(\varepsilon, t(\theta) = \pi/2)$ obtained through the solution of (6.9a) tend to a limit $R_{L,f}(0)$ as $\varepsilon \to 0$. The latter is the asymptotic value of the solution obtained by setting formally $\varepsilon = 0$ in (6.9a),(6.9b), i.e. with the replacements in $k(\theta)$ indicated in Lemma 6.1:

$$\begin{aligned}\frac{dR_{L,0}}{d\theta} =& \frac{7}{216}\frac{R_{L,0}(\theta)}{\tau(\theta)^{8/3}}\sin(2z_0) + \frac{R_{L,0}^2(\theta)}{4\tau(\theta)^{1/2}}(\sin z_0 + \sin 3z_0) \\ &+ \frac{R_{L,0}(\theta)^3}{12\tau(\theta)}(\sin 2z_0 + \frac{\sin 4z_0}{2})\end{aligned} \tag{D.1a}$$

$$\begin{aligned}\frac{d\phi_{L,0}}{d\theta} =& \frac{7}{216\tau(\theta)^{8/3}}(1 + \cos 2z) + \frac{R_{L,0}(\theta)}{4\tau(\theta)^{1/2}}(3\cos z_0 + \cos 3z_0) \\ &+ \frac{1}{12}\frac{R_{L,0}(\theta)^2}{\tau(\theta)}(\frac{3}{2} + 2\cos 2z_0 + \frac{1}{2}\cos 4z_0)\end{aligned} \tag{D.1b}$$

where we have used the limiting forms (see eqns. (4.10), (6.7)) $g(\theta) \approx (7/108)\tau(\theta)^{-8/3}$, $k(\theta) \approx 1/\tau(\theta)^{1/2}$ and $z_0 = \theta + \phi_{L,0}(\theta)$. In (D.1a), (D.1b), one uses $\tau(\theta)$ as determined from :

$$\theta(\tau) = \frac{3\sqrt{3}}{4}(\tau^{4/3} - \tau_0^{4/3}) \approx \frac{3\sqrt{3}}{4}\tau^{4/3} \tag{D.2}$$

where the last approximation is true for large τ. In this domain,

$$\tau(\theta) \approx \left(\frac{4\theta}{3\sqrt{3}}\right)^{3/4} \tag{D.3}$$

We perform now on equations (D.1a),(D.1b) the same "‘averaging"’ transformations as in Sect.5 and "‘eliminate"’ successively terms in $1/\theta^{3/8}$ and $1/\theta^{3/4}$. At this latter stage, there appear in the equation for $\phi_{L,0}$ secular terms, which may diverge as $\theta \to \infty$. It is convenient for the following to continue this procedure and eliminate further terms in $1/\theta^{9/8}$ at which stage it becomes apparent that further secular terms for $\phi_{L,0}$ occur to $O(1/\theta^{3/2})$, but no such terms occur for $R_{L,0}$. A further transformation to remove terms in $dk/d\theta \approx 1/\theta^{11/8}$ leads also in the equation for $R_{L,0}$ to secular terms of $O(kdk/d\theta) = O(1/\theta^{7/4})$. None of these terms is divergent as $\theta \to \infty$. They contribute finite quantities to the phase $\phi_{L,0}$. If we denote by $R_{4L,0}(\theta), \phi_{4L,0}(\theta)$ the dependent variables obtained after these four transformations, we find that these obey equations like:

$$\frac{dR_{4L,0}}{d\theta} = O(\theta^{-3/2}) \tag{D.4a}$$

$$\frac{d\phi_{4L,0}}{d\theta} = -\frac{7}{24}\left(\frac{3\sqrt{3}}{4}\right)^{3/4} \frac{R_{L,0}(\theta)^2}{\theta^{3/4}} + O(\theta^{-3/2}) \tag{D.4b}$$

In writing these equations, we have used the fact that, according to Sect.4, $R_{L,0}$ is bounded for all θ and that all other terms, coming from $g(\theta)$ and the derivatives of $k(\theta)$ fall off even quicker with θ. Integrating (D.4a) between two values θ_1 and θ_2 we conclude that:

$$|R_{4L,0}(\theta_2) - R_{4L,0}(\theta_1)| \leq const(\theta_1^{-1/2} - \theta_2^{-1/2}) \tag{D.5}$$

Since for a sequence of points $\theta_n \to \infty$ the differences between any two terms $R_{4L,0}(\theta_n)$, $R_{4L,0}(\theta_m)$ tends to zero when $n, m \to \infty$ it follows that the sequence $R_{4L,0}(\theta_n)$ approaches a limit, which we call $R_{L,f}(0)$. We return now step by step to the original variable $R_{L,0}(\theta)$ by inverting the transformations leading to $R_{4L,0}$; these transformations are such that the differences between $R_{iL,0}$ and $R_{jL,0}$ decrease to zero as $\theta \to \infty$ so that we conclude that $R_{L,0}(\theta)$ approaches the same limit $R_{L,f}(0)$.

Further, for a finite value of ε, for which $\theta(t = \pi/2) = const/\sqrt{\varepsilon}$, we perform the same transformations to obtain equations similar to (D.4a) and (D.4b), with the difference that on the right hand side we have powers of $k(\theta)$, and its product with derivatives or with $g(\theta)$:

$$\frac{dR_{4L,\varepsilon}}{d\theta} = O(k^4 + g) \tag{D.6a}$$

$$\frac{d\phi_{4L,\varepsilon}}{d\theta} = -\frac{7}{24}\{R_{L,\varepsilon}(\theta)^2 k(\theta)^2 + O(k^4 + g) \tag{D.6b}$$

It is necessary to keep the terms with $g(\theta)$ because $k(\theta)$ falls off exponentially for time scales of $O(1)$ and thus may become smaller than $g(\theta)$, which settles to a value of $O(\varepsilon(\ln(1/\varepsilon))^2$ after falling off like $O(\theta^{-2})$ when $t = o(1)$. Integrating (D.6a) between a sufficiently large value of θ and $\theta_f \equiv \theta(t = \pi/2)$ we obtain:

$$|R_{4L,\varepsilon}(\theta_f) - R_{4L,\varepsilon}(\theta)| \leq const \int_\theta^{\theta_f} (k(\theta)^4 + g(\theta))d\theta \tag{D.7}$$

If we choose $|\tau| > \varepsilon^{-\delta}$ for $0 < \delta < 3/8$, the integrals in eqn.(D.7) are less than $const \times \varepsilon^{2\delta/3}$; if $\delta = 3/8$ (i.e.$t = O(1)$), they are even less than $const\sqrt{\varepsilon}(\ln(1/\varepsilon))^2$ (the terms with $g(\theta)$ are dominant).

Thus, we must now show that the difference of the solutions of the limiting equations (D.4a) and (D.4b) to those of the exact equations (D.6a) and (D.6b) at such values of θ vanishes as $\varepsilon \to 0$. To this end, we write out (D.4a), (D.4b), (D.6a), (D.6b) in more detail (the precise values of the coefficients may be obtained from an algebraic manipulation program, but they are of no importance for the present purpose). For instance, (D.6a) reads :

$$\frac{dR_{4L,\varepsilon}}{d\theta} = k(\theta)^4 R_{L,\varepsilon}(\theta)^5 P_1(z) + ... \tag{D.8}$$

where $P_1(z)$ is a trigonometric polynomial of $z = \theta + \phi_{L,\varepsilon}(\theta)$. The further terms contain higher powers of $k(\theta)$ and products of $k(\theta)$ or its derivatives with $g(\theta)$. Equation (D.4a) is similar: the same polynomials in z occur, but (as in (D.1a),(D.1b)),$X_R(t)$ is replaced by the leading term of the inner expansion (3.23), i.e. for large θ, $k(\theta) \approx const/\theta^{3/8}$. From eqns.(3.9),(3.25) we see that, for large τ, the difference:

$$\eta_R(\tau) - \tau^{1/3} = O\left(\varepsilon^{3/4}\tau^{7/3}, \frac{\gamma}{\tau^{4/3}}\right) \tag{D.9}$$

Equation (D.9) allows us to estimate the differences between the various coefficients of eqn.(D.8) and its analogon (D.4a). We notice that the differences have to be estimated at a fixed value of θ, which corresponds to different values of τ denoted by $\tau(0,\theta)$, $\tau(\varepsilon,\theta)$ in turn, in the situations $\varepsilon = 0$ and finite ε (cf.also Appendix C, eq.(C.13)). The value of τ corresponding to θ at $\varepsilon = 0$ is given by (D.3) but for finite ε it is the solution of

$$\theta \cong \sqrt{3}\int_{\tau_0}^{\tau} \eta_R(\tau\prime)d\tau\prime \cong \frac{3\sqrt{3}}{4}\tau^{4/3}(1 + O(\varepsilon^{3/4}\tau^2)) \tag{D.10}$$

where we have used the estimate (D.9) for $\eta_R(\tau)$. For small ε and $\tau < \varepsilon^{-\delta}$, $\delta < 3/8$,one obtains:

$$\tau(\varepsilon,\theta) = \tau(0,\theta)(1 + O(\varepsilon^{3/4}\tau(0,\theta)^2)) \tag{D.11}$$

Using this in (D.9) one gets an estimate:

$$\eta_R(\tau(\varepsilon,\theta)) = \tau(0,\theta)^{1/3}(1 + O(\varepsilon^{3/4}\tau(0,\theta)^2)) \tag{D.12}$$

which shows that in evaluating orders of magnitudes of differences at fixed θ one may replace $\tau(\varepsilon,\theta)$ by $\tau(0,\theta)$ of (D.3). Let now:

$$\Delta R_4(\theta) \equiv R_{4L,\varepsilon}(\theta) - R_{4L,0}(\theta) \qquad \Delta R(\theta) \equiv R_{L,\varepsilon}(\theta) - R_{L,0}(\theta) \tag{D.13a}$$

$$\bar{\phi}_{4L,\varepsilon}(\theta) \equiv \phi_{4L,\varepsilon}(\theta) + \frac{7}{24}\int_0^{\theta} k(\theta)^2 R_L(\theta)^2 d\theta \tag{D.13b}$$

$$\Delta\phi_4(\theta) \equiv \phi_{4L,\varepsilon}(\theta) - \phi_{4L,0}(\theta) \qquad \Delta\phi(\theta) \equiv \phi_{L,\varepsilon}(\theta) - \phi_{L,0}(\theta) \tag{D.13c}$$

and $\bar{\phi}_{L,\varepsilon}$ be related to $\phi_{L,\varepsilon}$ by (6.12). Subtracting from each other equations like (D.8) and their counterparts for $\varepsilon = 0$ and using the uniform boundedness with respect to ε of $R_L(\theta)$ one obtains

$$\frac{d\Delta R_{4L}}{d\theta} \leq const\Delta(k(\theta)^4) + a_1(\theta)\Delta R_L + b_1(\theta)\Delta\phi \tag{D.14a}$$

$$\frac{d\Delta\bar{\phi}_{4L}}{d\theta} \leq const\Delta(k(\theta)^4) + a_2(\theta)\Delta R_L + b_2(\theta)\Delta\phi \tag{D.14b}$$

$$\Delta\phi_4 \leq \Delta\bar{\phi}_4 + \frac{7}{24}\Delta\int_0^{\theta} k(\theta)^2 R_L(\theta)^2 d\theta \tag{D.14c}$$

In equations (D.14a), (D.14b), the functions $a_i(\theta), b_i(\theta)$ are sums over the various $\theta-$ dependent coefficients in (D.6a),(D.6b) multiplied by constants obtained from the simple estimates $|P_i(z)| < const$. As long as t is confined to an interval $(\varepsilon^{3/8} < t < C_\varepsilon), C_\varepsilon \to 0$ as $\varepsilon \to 0$, the dominant term is of $O(k^4) = O(\theta^{-3/2})$, so that we may assume this is the order of magnitude of the $a_i(\theta), b_i(\theta)$ in (D.14a), (D.14b). From a repeated application of the "‘inverse"’ eqns. (5.9a),(5.9b) we may express ΔR_L, $\Delta\phi_L$ in terms of ΔR_{4L},$\Delta\phi_{4L}$ through:

$$\Delta R_L = \Delta R_{4L}(1 + O(k(\theta))) + \Delta\phi_{4L}O(k(\theta)) + O(\Delta k(\theta)) \tag{D.15a}$$

$$\Delta\phi_L = \Delta R_{4L}O(k(\theta)) + \Delta\phi_{4L}(1 + O(k(\theta))) + O(\Delta k(\theta)) \tag{D.15b}$$

Upon substitution in (D.14a),(D.14b), one verifies that the estimates remain unchanged if we replace ΔR_L, $\Delta\phi_L$ by ΔR_{4L}, $\Delta\phi_{4L}$ and also the order of magnitude of the coefficients is preserved (we keep the notation unchanged). To replace further $\Delta\phi_{4L}$ by $\Delta\bar{\phi}_{4L}$ we need a bound on the integral in (D.14c). If M is a bound on R_L, R_{L0} we obtain, using (D.15a),(D.15b):

$$\begin{aligned}
&\int_0^\theta \Delta(k^2R_L)^2 d\theta\prime \leq M^2\int_0^\theta (\Delta k)^2 d(\theta\prime) + 2M\int_0^\theta k^2\Delta R_L(\theta\prime)d\theta\prime \\
&\leq M^2\int_0^\theta (\Delta k)^2 d(\theta\prime) + C_1\int_0^\theta k^2\Delta R_{4L}d\theta\prime + C_2\int_0^\theta k^3\Delta\bar{\phi}_{4L}d\theta\prime + \\
&C_3\int_0^\theta k(\theta\prime)^3 d\theta\prime\int_0^{\theta\prime}\Delta(k^2R_L^2)d\theta\prime\prime
\end{aligned} \tag{D.16}$$

with C_1, C_2, C_3 constants pertaining to the $O()$ terms in (D.15a),(D.15b). If we invert the order of integration in the last term and realize that, for a sufficiently high τ_0 it is true that, for all $0 < \theta\prime < \theta$:

$$1 - C_3\int_{\theta\prime}^\theta k(\theta\prime\prime)^3 d\theta\prime\prime \geq 1 - C_3\int_0^\theta k^3 d\theta\prime\prime \geq 1 - \frac{C_3}{\tau_0^{1/6}} > const > 0 \tag{D.17}$$

we may conclude that:

$$\int_0^\theta \Delta(k^2R_L^2)d\theta\prime \leq const\left(\int_0^\theta (\Delta k)^2 d\theta\prime + \int_0^\theta k^2\Delta R_{4L}d\theta\prime + \int_0^\theta k^3\Delta\bar{\phi}_{4L}d\theta\prime\right) \tag{D.18}$$

This allows us to obtain inequalities like (D.14a), (D.14b) only in terms of R_{4L}, $\bar{\phi}_{4L}$. We integrate now the resulting inequality in (D.14a) from $\theta = 0$ to θ and interchange the order of integration. One obtains:

$$|\Delta R_{4L}(\theta)| \leq |\Delta R_{4L}(0)| + T(\theta) + \int_0^\theta c_1(\theta,\theta\prime)\Delta R_{4L}(\theta\prime)d\theta\prime + \int_0^\theta c_2(\theta,\theta\prime)\Delta\bar{\phi}_{4L}(\theta\prime)d\theta\prime \tag{D.19}$$

where the dominant terms are now originating in (D.16): e.g.using the notation in (D.14a)

$$T(\theta) = \int_0^\theta \Delta(k^2)(\theta\prime)d\theta\prime\int_{\theta\prime}^\theta b_1(\theta\prime\prime)d\theta\prime\prime = O(\varepsilon^{3/4}\theta^{5/4}) \tag{D.20}$$

and

$$|c_1(\theta,\theta\prime)| < k(\theta\prime)^2\int_{\theta\prime}^\theta a_1(\theta\prime\prime)d\theta\prime\prime = O(\theta\prime^{-5/4}) \tag{D.21}$$

In (D.20) we have used the estimate (D.12) of $\eta_R(\tau)$:

$$\Delta k(\theta)^2 = k(\theta)^2 - \frac{1}{(\tau(\theta)^{1/2})^2} = const\frac{1}{\tau(\theta)}\left(\frac{1}{(1+\varepsilon^{3/4}\tau^2)^3} - 1\right) = O(\varepsilon^{3/4}\tau) \tag{D.22}$$

Estimates similar to (D.21) hold for $c_2(\theta,\theta\prime)$ and for the coefficients appearing in the inequality for $\Delta\bar{\phi}_4$ analogous to (D.19). Defining:

$$\Delta\sigma(\theta) \equiv |\Delta R_{4L}(\theta)| + M|\Delta\bar{\phi}_{4L}(\theta)| \tag{D.23}$$

and adding (D.19) and its analogon for $\Delta\bar{\phi}_{4L}$ one obtains:

$$\Delta\sigma(\theta) \leq T_1(\theta) + \int_0^\theta d(\theta\prime)\Delta\sigma(\theta\prime)d\theta\prime \tag{D.24}$$

where $T_1(\theta)$ obeys an estimate like (D.20) and $d(\theta)$ an estimate like (D.21). Gronwall's inequality implies then:

$$\Delta\sigma(\theta) = O(\varepsilon^{3/4}\theta^{5/4}) \tag{D.25}$$

and thus vanishes as $\varepsilon \to 0$ on intervals of θ of order $\varepsilon^{-4\delta/3}$ with $\delta < 3/8$. With this, the difference between the value $R_{4L,\varepsilon}(t = \pi/2) \equiv R_{4L,f}(\varepsilon)$ and the asymptotic value $R_{4L,f}(0)$ obtained from the equation with $\varepsilon = 0$ is:

$$|R_{4L,f}(\varepsilon) - R_{4L,f}(0)| \leq |R_{4L,f}(\varepsilon) - R_{4L,\varepsilon}(\theta)| + |R_{4L,\varepsilon}(\theta) - R_{4L,0}(\theta)| + |R_{4L,0}(\theta) - R_{4,f}(0)| \tag{D.26}$$

For $\theta = O(\varepsilon^{-4\delta/3})$,$\delta < 3/8$, the first term is $O(\varepsilon^{2\delta/3})$ (cf.the comments on (D.7)), the second term is $O(\varepsilon^{3/4-5\delta/3})$ (cf. eqn. (D.25)) and the third term is also of $O(\theta^{-1/2}) = O(\varepsilon^{2\delta/3})$. For any choice of $\delta < 3/8$, the difference (D.26) vanishes as $\varepsilon \to 0$. Inverting the transformations leading from $R_L(\theta)$ to $R_{4L}(\theta)$, this result holds also for the original variables $R_L, R_{L,0}$, because $k(\theta)$ approaches $k(\varepsilon = 0, \theta) \equiv 1/\tau(\theta)^{1/2}$ as $\varepsilon \to 0$ on the whole τ-interval $[1, \pi/2/\varepsilon^{3/8}]$. We obtain thus the statement of Lemma 6.1. From this argument it follows that the asymptotic value of $R_L(\theta)$ is approached closely even in the "transition domain" between the boundary layer region and that where $t = O(1)$. This limiting value for R_L may be obtained once for all by solving the boundary layer equation for $X_L(t)$ for t>0 using for $X_R(t)$ the simple approximation $X_R(t) \approx \varepsilon^{1/8}\eta_{00}(\tau)$ and looking at the asymptotic value of $R_L(\theta)$. It turns out to be ≈ 0.844 Clearly, the same argument as above serves to show that the phase difference $\Delta\bar{\phi}_{4L}(\theta)$ tends to zero as $\varepsilon \to 0$ and thus justify the contents of Lemma 6.3.The limiting value $\bar{\phi}_{L,f}(0)$ depends on the choice of τ_0 (the origin of the variable θ_R; for $\tau_0 = 15$,$\bar{\phi}_{L,f}(0) \approx 0.14$)

E. The Evaluation of some Integrals

In this Appendix we drop for simplicity the index L used in Section 6:$R(\theta) \equiv R_{L,\varepsilon}(\theta)$*, etc.* To evaluate (6.50) we add and subtract $R(\theta(\pi/2))(\partial R/\partial R_0)(\theta)$ and we are thus led to the evaluation of the integrals:

$$\mathcal{I}_1 \equiv \int_0^{\theta(\pi/2)} k(\theta)^2 \frac{\partial R}{\partial R_0}(\theta)(R(\theta) - R(\theta(\pi/2))d\theta \tag{E.1}$$

$$\mathcal{I}_2 \equiv \int_0^{\theta(\pi/2)} k(\theta)^2 R(\theta) \left[\left(\frac{\partial R}{\partial R_0}\right)(\theta) - \left(\frac{\partial R}{\partial R_0}\right)(\theta(\pi/2))\right] d\theta \tag{E.2}$$

For $\mathcal{I}_1$ we use the boundedness of $\partial R/\partial R_0$ and the relation:

$$R(\theta) = R_4(\theta) + O(k(\theta)) \tag{E.3}$$

obtained by iteration of (5.9a),(5.9b) to obtain:

$$\begin{aligned} \mathcal{I}_1 &< const \int_0^{\theta(\pi/2)} k(\theta)^2(|R_4(\theta) - R_4(\theta(\pi/2))| + O(k(\theta))d\theta \\ &< const(\int_0^{\theta(\pi/2)} k(\theta)^3 d\theta) + O\left(\int_0^{\theta(\pi/2)} k(\theta\prime)^2 d\theta\prime \int_{\theta\prime}^{\theta(\pi/2)} k(\theta\prime\prime)^4 d\theta\prime\prime)\right) \end{aligned} \tag{E.4}$$

where we have used (D.7) to evaluate the difference $|R_4(\theta) - R_4(\theta(\pi/2))|$ and the fact that $\delta R(\theta)$ is bounded (see text following eqn.(6.31)). Now, both integrals in (E.4) are bounded so that we deduce:

$$\mathcal{I}_1 < \infty \tag{E.5}$$

Using Lemma 6.4 we may also conclude that $\mathcal{I}_1$ has a limit as $\varepsilon \to 0$.

For $\mathcal{I}_2$ we write, using (6.24a):

$$\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22} \tag{E.6}$$

$$\mathcal{I}_{21} = \int_0^{\theta(\pi/2)} k(\theta)^2 R(\theta) \left[\left(\frac{\partial R_4}{\partial R_0} \right) (\theta)(1 + O(k)) - \left(\frac{\partial R_4}{\partial R_0} \right) (\theta(\pi/2))(1 + O(k(\pi/2))) \right] d\theta \tag{E.7}$$

$$\begin{aligned} \mathcal{I}_{22} &\equiv \int_0^{\theta(\pi/2)} k(\theta)^2 R(\theta) \left[\left(\frac{\partial \phi_4}{\partial R_0}(\theta) \right) O(k(\theta)) - \left(\frac{\partial \phi_4}{\partial R_0}(\theta(\pi/2)) \right) O(k(\theta(\pi/2)) \right] d\theta \\ &\equiv \mathcal{I}_{221} + \mathcal{I}_{222} \end{aligned} \tag{E.8}$$

In (E.7) we replace the difference of the values of $\partial R_4/\partial R_0$ at θ and $\theta(\pi/2)$ by the integral over the right hand side of (6.23a). Since both $R(\theta)$ and $\partial R/\partial R_0(\theta)$ are bounded, the integral over the first term in (6.23a) converges. The second term is more difficult, since $\partial\phi/\partial R_0$ diverges like $\theta^{1/4}$, so that the integral from θ to ∞ falls off like $1/\theta^{1/4}$; this brings a contribution behaving like $\ln(1/\gamma)$ in $\mathcal{I}_{21}$. To improve on this, we use below the fact that the polynomial $\bar{P}_2(z)$ has *zero* mean. We expect namely that the oscillations of $\bar{P}_2(z)$ reduce the magnitude of the integrals. Before proceeding, we move to $\mathcal{I}_{22}$, where a similar problem occurs. The second term of the integral in (E.8) is negligible, since $k(\theta(\pi/2))$ is exponentially small. In the first term, the factor $\partial\phi_4/\partial R_0$ increases like $\theta^{1/4}$ (see eqn.(6.37)) so that the power of $1/\theta$ under the integral is at a first sight 7/8, which is not enough for convergence. However, as remarked in relation to eqns.(6.24a),(6.24b), the true appearance of this term is (we leave out the last term):

$$\mathcal{I}_{221} = \int_0^{\theta(\pi/2)} k(\theta)^3 R(\theta)^2 \frac{\partial \phi_4}{\partial R_0}(\theta) T(z_4) d\theta \tag{E.9a}$$

$$z_4 = \theta + \phi_4(\theta) \tag{E.9b}$$

where $T(z_4)$ is a trigonometric polynomial with zero mean (in this case, $T(z_4) = \sin z_4 + (1/3)\sin(3z_4)$, cf.eq.(5.9a),(5.9b). The property of zero mean ensures that there exists another trigonometric polynomial (which may be also chosen to have zero mean)$S(z_4)$ so that:

$$\frac{dS}{dz_4} = T(z_4) \tag{E.10}$$

We transform then $\mathcal{I}_{221}$ by partial integration:

$$\begin{aligned} \mathcal{I}_{221} &= k(\theta)^3 R(\theta)^2 \frac{\partial \phi_4}{\partial R_0} \frac{1}{1 + d\phi_4/d\theta} S(z_4)\Big|_0^{\theta(\pi/2)} \\ &- \int_0^{\theta(\pi/2)} \frac{d}{d\theta} \left[k^3(\theta) R^2(\theta) \frac{\partial \phi_4}{\partial R_0} \frac{1}{1 + d\phi_4/d\theta} \right] S(z_4) d\theta \end{aligned} \tag{E.11}$$

Now,the derivatives with respect to θ in eq.(E.11) have as effect the increase of the rate of falloff of the integrand with respect to θ: indeed,

$$\frac{dR}{d\theta} = O(k), \qquad \frac{d}{d\theta} \frac{\partial \phi_4}{\partial R_0} = O(k^2)$$

as follows from eqn.(6.9a) and (6.23b), and

$$\frac{d^2 \phi_4}{d\theta^2} = O(k^2)$$

as follows from eq.(D.6b). Further, the denominator $1 + d\phi_4/d\theta$ is nonvanishing if only we take the starting point τ_0 sufficiently large. As a consequence, the integrals containing each of these derivatives are absolutely convergent. The first term is clearly bounded, so that we conclude that $\mathcal{I}_{22}$ is itself bounded. The same argument may be applied to the second term in $\mathcal{I}_{21}$: a partial integration increases the falloff rate of the integral from θ to ∞ and ensures thus the boundedness of $\mathcal{I}_{21}$. Moreover, after performing the partial integration, it is possible to take the limit $\varepsilon \to 0$. Indeed all integrals can be seen to make sense and be finite even if we set in them formally $\varepsilon = 0$ and replace all quantities with those with superscript 0 (corresponding to $\varepsilon = 0$,see Section 6.2). Moreover, the values of the integrals tend to those for $\varepsilon = 0$ as $\varepsilon \to 0$. To see this, one notices that: (i)since all integrals extended from 0 to ∞ are absolutely convergent, the contribution of the interval $(\varepsilon^{-s}, \theta(\pi/2))$, $s > 0$ may be made arbitrarily small, for ε small enough.(ii)The departure of the various quantities of interest $(R(\theta), \partial R/\partial R_0(\theta), etc.)$ from their values for $\varepsilon = 0$ (indexed with a superscript "'0"' in Section 6.2) is estimated by expressions like $\varepsilon^a \theta^b$ with $a > 0, b \geq 0$(see eqns.(6.44),(D.25)).Their integrals over intervals $(0, \varepsilon^{-s})$ are quantities of $O(\varepsilon^a \varepsilon^{-(b+1)s})$. This is a positive power of ε if $s < a/(b+1)$. This shows that, indeed, all integrals are continuous at $\varepsilon = 0$

F. On the Invariant Sets of the Circle Map Π

In this Appendix a proof is given for Lemma 7.2.

F.1. *General Comments*

(i)We restrict ourselves to $0 < \beta < \pi$; it follows we can assume $-\pi < \chi < \pi$. It is of some advantage to assume $-\pi/2 < \Sigma < 3\pi/2$ and to write:

$$\Sigma = \pi/2 + \hat{\Sigma}, \qquad \Pi(\chi) = -\beta \sin(\chi + \tilde{\Sigma}) \tag{F.1}$$

With this, eqns. (7.25),(7.26),(7.27) for β_{2u},β_{2d} are changed to the unified form :

$$\beta_{2u} \sin(\beta_{2u} + \hat{\Sigma}) = \pi/2 + \hat{\Sigma}(mod(2\pi)), \qquad \beta_{2u} \neq \pi/2 - \hat{\Sigma} \equiv \beta_{1u}(mod(2\pi)) \tag{F.2}$$

$$\beta_{2d} \sin(\beta_{2d} - \hat{\Sigma}) = \pi/2 - \hat{\Sigma}(mod(2\pi)), \qquad \beta_{2d} \neq \pi/2 - \hat{\Sigma} \equiv \beta_{1d}(mod(2\pi)) \tag{F.3}$$

The solutions β_{2u}, β_{2d} correspond to ("'superstable"') period two orbits passing through the maxima (at $\chi_M = -\pi/2 - \hat{\Sigma}$) and minima(at $\chi_m = \pi/2 - \hat{\Sigma}$) of $\Pi(\chi)$, eq.(F.1)[37]. The specification $(mod 2\pi)$ means for β_{2u}, β_{2d} that the quantity must be transferred to the interval $(-\pi, \pi)$ through suitable addition or subtraction of 2π. The values of β_{1u}, β_{1d} are also solutions of the equations satisfied by β_{2u}, β_{2d} and immediately precede the latter. They play a role only if they are positive and less than π (after 2π translation). Eqns.(F.2),(F.3) show the symmetry:

$$\beta_{2u}(-\hat{\Sigma}) = \beta_{2d}(\hat{\Sigma}) \tag{F.4}$$

In Fig.14 we show the appearance of the solutions $\beta_{2u,2d}(\Sigma)$ in the $\Sigma - (-\beta)$ plane of Fig.11. The right hand branch (with respect to $\hat{\Sigma} = 0$)corresponds to $\beta_{2d}(\Sigma)$, the left hand one (small squares)to $\beta_{2u}(\Sigma)$.

(ii)For any $\alpha \in (-\beta, \beta)$, the equation $\Pi(\beta, \hat{\Sigma}; \chi) = \alpha$ has two solutions with $-\pi < \chi < \pi$. If $\alpha = \chi_+(\chi_-)$ is a fixed point of Π, we denote its pair by $\tilde{\chi}_+(\tilde{\chi}_-)$ (If the fixed point is positive, it is denoted by $\chi_+ > 0$, otherwise by $\chi_- < 0$).

(iii)According to (7.13), $S(\Pi) < 0$. This has the consequences (see Collet & Eckmann [1983, p.97]):(a)$S(\Pi^p) < 0$ [38] for all p>0; (b)$|d(\Pi^p)/d\chi|$ cannot have a strictly positive minimum; (c)if $d\Pi^p/d\chi$ does not change sign for $\chi \in [a, b]$ and Π^p has three fixed points there, the middle one is unstable and the other two are stable; (d)Π^p cannot have more than three fixed points in an interval $[a, b]$ where $d\Pi^p/d\chi > 0 (< 0)$.

[37] At $\beta = \beta_{1u}(\beta_{1d})$ the fixed point lies at the maximum (minimum) of $\Pi(\chi)$
[38] $\Pi \circ \Pi \circ ... \circ \Pi$ (p times)$\equiv \Pi^p$

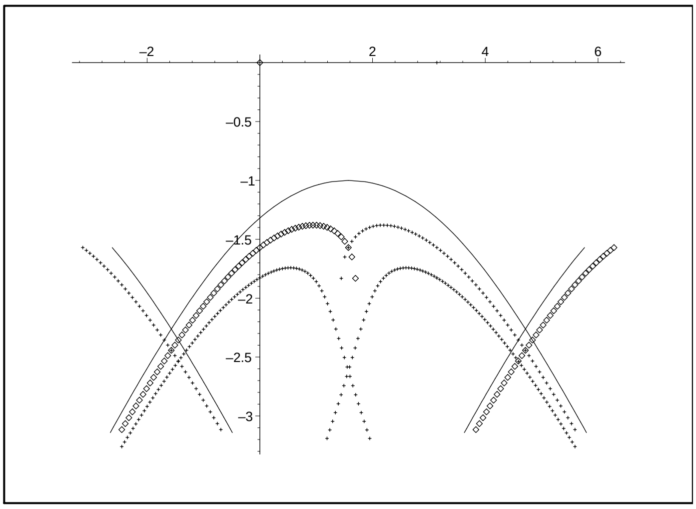

Figure 14: The β-values for superstable orbits

F.2. *The situation* $\hat{\Sigma} = -\pi$

At $\hat{\Sigma} = -\pi$ $(\Sigma = -\pi/2)$, Π may be decomposed into two maps Π_+, Π_- of $[-\beta, 0]$, $[0, \beta]$ into themselves.From (F.2), (F.3) one verifies that $\beta_{1u} = \beta_{1d} = \pi/2$, $\pi/2 < \beta_{2u}, \beta_{2d} < \pi$.

If $\beta < 1$, only $\chi = 0$ is a fixed point and Π is a contraction (Lemma 7.1). If $\beta > 1$, there are three fixed points, χ_-, $\chi_0 = 0$, χ_+; χ_0 is unstable. If $\beta \leq \pi/2$, Π_+ contracts the interval $[0, \chi_+]$ into itself and to χ_+. Indeed, on $[0, \chi_+]$,(i)$\Pi_+(\chi)$ is monotonically increasing,(ii)$\Pi_+(\chi) - \chi > 0$ and vanishes only at the ends,(iii)$\Pi_+(\chi) < \chi_+$ since $0 = \Pi_+(\chi_+) - \chi_+ > \Pi_+(\chi) - \chi_+$. It follows that, for any $\chi \in [0, \chi_+]$ the monotonically increasing sequence $\{\chi, \Pi(\chi), \Pi^2(\chi)...\}$ has a limit which can only be χ_+. Further, the interval $[\chi_+, \beta]$ is mapped into itself and (after iterations) to χ_+. Indeed, for any χ in this interval, $\chi_+ < \Pi_+(\chi) < \chi$ so that the sequence $\{\Pi^p(\chi)\}$ converges to χ_+. If $\beta = \pi/2$, $\chi_+ = \pi/2$.

Consider next the interval $\pi/2 < \beta < \beta_F(\hat{\Sigma} = -\pi)$ (see Fig.14).Now $\chi_+ > \pi/2$ and $\tilde{\chi}_+ < \pi/2$. The iterates of any point in $[0, \tilde{\chi}_+]$ reach at some stage the interval $[\tilde{\chi}_+, \chi_+]$. Indeed, the function $\Pi_+(\chi)$ is monotonically increasing on $[0, \tilde{\chi}_+]$ so that the sequence $\{\chi, \Pi_+(\chi)..\Pi_+^p(\chi)\}$ increases monotonically until $\Pi_+^p(\chi)$ gets larger than $\tilde{\chi_+}$. Then the following transformations under Π_+ are obvious:

$$[\tilde{\chi}_+, \chi_+] \Rightarrow [\chi_+, \beta] \Rightarrow [\Pi_+(\beta), \chi_+]. \tag{F.5}$$

Now, for $\beta < \beta_{2u}(\hat{\Sigma} = -\Pi)$,

$$\Pi_+(\beta) \equiv \beta \sin \beta > \pi/2 \tag{F.6}$$

(cf.(F.2)) so that the last interval is contained in $[\pi/2, \chi_+]$.Thus Π_+^2 maps $[\pi/2, \chi_+]$ into itself. The same is true for $[\chi_+, \beta]$ which is mapped by Π_+ into $[\pi/2, \chi_+]$ and by Π_+^2 into itself (cf.(F.5)). It is also true that, for $\chi \in [\pi/2, \beta]$:

$$(\Pi_+^2)\prime(\chi) = \Pi_+\prime(\Pi_+(\chi))\Pi_+\prime(\chi) > 0; \tag{F.7}$$

because $\beta < \pi$. This inequality is true for all $\beta \in [\pi/2, \beta_{2u}]$. It follows from the property $S(\Pi) < 0$, (7.13) that Π_+^2 has at most three fixed points in $[\pi/2, \beta]$.

We show now that, if $\beta < \beta_F(< \beta_{2u})$ the only invariant set of Π_+^2 in $[\pi/2, \beta]$ is $\{\chi_+\}$. Indeed, χ_+ is the only root of $\Pi_+^2(\chi) = \chi$ in $[\pi/2, \beta]$ in this interval of values of β. For $\chi = \pi/2$, $\Pi_+^2(\chi) > \chi$ and for $\chi = \beta$, $\Pi_-^2(\chi) < \chi$. Thus, for all $\chi \in [\pi/2, \chi_+]$, $\Pi_+^2(\chi) > \chi$ and for all $\chi \in [\chi_+, \beta]$, $\Pi_+^2(\chi) < \chi$. It follows that for $\chi \in [\pi/2, \chi_+]$ the sequence $\left\{\chi, \Pi_+^2(\chi)....\right\}$ is monotonically increasing and can only converge to χ_+. Similarly $\left\{\chi, \Pi_+^2(\chi)...\right\}$ is monotonically decreasing for $\chi \in [\chi_+, \beta]$ and converges to χ_+. Thus the only invariant set is indeed $\{\chi_+\}$.

We now turn to the situation $\beta_F < \beta < \beta_{2u}$. There are now three points (χ_L, χ_+, χ_R) in $[\pi/2, \beta]$ where $\Pi_+^2(\chi) - \chi$ changes sign. Consequently, the sequences $\{\chi, \Pi_+^2(\chi)....\}$ converge (monotonically increasing or decreasing) to χ_L or χ_R according to whether they start in $[\pi/2, \chi_+]$ or in $[\chi_+, \beta]$. Now, the image $\chi_{L'}$ of χ_L under Π_+ must be a fixed point of Π_+^2 because $\Pi_+^3(\chi_L) = \Pi_+^2(\chi_{L'}) = \Pi_+(\chi_L) = \chi_{L'}$ so that $\chi_{L'} \equiv \chi_R$. Thus χ_+ and an orbit of period two exhaust the invariant sets of Π_+ for $\beta < \beta_{2u}$. The discussion of Π_- is totally symmetric to the above with β_{2u} replaced by β_{2d}.

To conclude, if $\hat{\Sigma} = -\pi$, $\beta_{2u} = \beta_{2d}$ and in the interval $0 < \beta < \beta_{2u}$ the mapping Π has no other invariant sets apart from at most three fixed points and two orbits of period two.

F.3. *The general situation*

The arguments concerning the contraction of intervals to the fixed points of Π and Π^2 may all be taken over from the situation $\hat{\Sigma} = -\pi$ of the previous section. We describe only the main features.

(i)Assume $\hat{\Sigma} = -\pi + \delta$, $\delta < \pi/2$. As $\delta \to \pi/2$,$\beta_{1u}(\hat{\Sigma}) \to 0$, $\beta_{1d}(\hat{\Sigma}) \to \pi$, $\beta_{2u}(\hat{\Sigma}) \to \pi/2$, $\beta_{2u}(\hat{\Sigma}) < \beta_{2d}(\hat{\Sigma})$ and β_S increases. (a)if $\beta < \beta_S(\hat{\Sigma})$ (cf.eq.(7.14)), there exists a single fixed point $\chi_+ > 0$; if further $\beta < \beta_{1u}(\hat{\Sigma}) \equiv \pi/2 - \delta$, then $\tilde{\chi}_+ < \chi_+$ and all points in $[-\beta, \beta]$ are attracted to χ_+. If $\beta_S(\hat{\Sigma}) > \beta > \beta_{1u}$, but $\beta < \beta_{2u}$, then $\tilde{\chi}_+ < \chi_+$ and under Π:

$$[-\beta, \beta] \Rightarrow [\tilde{\chi}_+, \chi_+] \Rightarrow [\chi_+, \beta] \Rightarrow [\Pi(\beta), \chi_+] \subset [\pi/2 - \delta, \chi_+]$$

since $\beta_{2u} \equiv \Pi(\beta) > \beta_{1u} = \pi/2 - \delta$. As before, this implies $(\Pi^2)\prime > 0$ on $(\pi/2 - \delta, \beta)$. For $\beta_F < \beta < \beta_{2u} < \beta_S$ there appear two further fixed points of Π^2, to which the intervals on the left and right of χ_+ are contracted under Π^2. (b)If $\beta > \beta_S(\hat{\Sigma})$ there are two further fixed points $\chi_- < \chi_0 < 0$, χ_- is stable, χ_0 unstable. The interval $[\chi_0, \tilde{\chi}_+]$ is mapped eventually into $[\tilde{\chi}_+, \chi_+]$ and the further evolution is the same as above. If $\beta < \beta_{1d} = \pi/2 + \delta$, $\tilde{\chi}_- < \chi_-$ and the interval (χ_-, χ_0) is contracted under Π to χ_-. If $\beta_{2d} > \beta > \beta_{1d}$, $\tilde{\chi_-} > \chi_-$ and (possibly after iteration)

$$[\tilde{\chi_-}, \chi_0] \Rightarrow [\chi_-, \tilde{\chi_-}] \Rightarrow [-\beta, \chi_-] \Rightarrow [\chi, \Pi(-\beta)] \subset [\chi_-, -\pi/2 - \delta].$$

The last inclusion follows from $\beta_{2d} > -\Pi(-\beta) > \beta_{1d}$. A consequence of this inclusion is that $(\Pi^2)\prime(\chi) > 0$ for $\chi \in [\chi_-, -\pi/2 - \delta]$, similarly to (F.7). If $\beta < \beta_F$, one concludes that the only invariant set in $[-\beta, \chi_0]$ is $\{\chi_-\}$. If $\beta > \beta_F$ a stable orbit of period two appears, but no other invariant sets. Thus, for $\hat{\Sigma} \in [-\pi, -\pi/2]$,the invariant set of Π consists of at most three fixed points and two orbits of period two. The invariant set depends on the value of $\hat{\Sigma}$ (as δ approaches $\pi/2$, the "‘lower" orbit of period two disappears).

(ii)If $\hat{\Sigma} \in [-\pi/2, 0]$, one verifies that $\beta_{2u}(\hat{\Sigma}) < \pi/2$ and $\beta_{2u}(-\pi/2) = \beta_{2u}(0) = \pi/2$ (cf. Fig. 14) ; also $\beta_{2u} \leq \beta_{2d}$ with equality at $\hat{\Sigma} = 0$. Further, $\beta_S(\hat{\Sigma}) > \pi/2$ so that there is no fixed point in $\chi < 0$. For all $\beta < \beta_{2u}$, there is only one fixed point at $\chi_+ > 0$. Let $\hat{\chi} \equiv \max[-\beta, \tilde{\chi_+}]$, $(\tilde{\chi_+} < 0)$. The image under Π of $[\hat{\chi}, \chi_+]$ is $[\chi_+, \beta]$. The latter is mapped back onto $(\Pi(\beta), \chi_+) \subset [-\pi/2 - \hat{\Sigma}, \chi_+]$. Since $\beta < \pi/2 - \hat{\Sigma}$ (the latter is the position of the minimum), it follows as before that $(\Pi^2)\prime > 0$ on $[-\pi/2 - \hat{\Sigma}, \beta]$. This interval may thus contain one or three fixed points of Π^2. Repeating the argument of the previous situation, we conclude that the invariant sets of Π consist of at most one fixed point and one orbit of period two.

(iii)If $\hat{\Sigma} \in [0, \pi/2]$, the situation is totally symmetrical to the previous one, with β_{2d} now interchanged with β_{2u} (cf.Fig.14) and the unique fixed point of Π being now situated at $\chi_- < 0$. The invariant set of Π consists of at most one fixed point and one 2-orbit.

(iv)If $\hat{\Sigma} \in [\pi/2, \pi]$, the situation is symmetrical (in the sense above) to that in (i), with the same conclusion. This ends the justification of Lemma 7.2

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