ON THE DUFFING EQUATION IN THE LIMIT OF LARGE FORCING AND LARGE DAMPING

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Abstract: We present a study of the periodic solutions of Duffing's equation:

$$\ddot{y} + 2\Delta \dot{y} + y^3 = \Gamma \cos t$$

for large values of the parameters Δ, Γ. We show that, if Δ increases monotonically as a function of Γ and

$$\lim_{\Gamma \to \infty} \ln \Gamma / \Delta(\Gamma) = 0$$

the equation admits of a unique periodic solution for Γ large enough. We present arguments that bifurcations occur if $\Delta(\Gamma) \sim \text{const} \ln \Gamma$.

PACS indices: 02.30+g, 02.60+y, 02.90+p

I. Introduction

We consider the Duffing equation:

$$\ddot{y} + 2\Delta \dot{y} + y^3 = \Gamma \cos t \qquad , \Delta, \Gamma > 0 \tag{1.1}$$

As is well known (see Refs.[1-3]), this equation may exhibit a large variety of periodic solutions, not necessarily of the same period as the driving force, and whose number and appearance changes with the values of the parameters Δ and Γ. According to numerical evidence (see e.g. Ref.[2], Fig. 1), the situation simplifies if the damping is large enough, at fixed Γ : the equation admits then of a unique periodic solution. We notice that, if (1.1) has only one periodic solution $y_p(t)$, then $y_p(t)$ has period 2π and its Fourier series contains only odd harmonics. Indeed, $y_{p1}(t) = -y_p(t+\pi)$ is also a periodic solution of (1.1) and it is, by assumption, identical to $y_p(t)$.

Now, with one exception (Ref.[4]), the literature appears to contain no statement describing the region in the plane of the parameters Γ, Δ where (1.1) admits of a unique periodic solution. In Ref.[4], W.S. Loud shows, using a result of Cartwright and Littlewood (Ref.[5]), that, if an harmonic term $+ky$ is present in (1.1), then (1.1) has a unique periodic solution at every fixed Γ, provided Δ is large enough (essentially $\Delta > \text{const} \cdot \Gamma$); the theorem of Ref.[5] is, however, not readily extensible to the situation $k = 0$.

In the present paper, we consider the case when both Δ, Γ are large; specifically, we assume Δ increases monotonically as a function of Γ and becomes unbounded as $\Gamma \to \infty$. Two problems may be raised in this connection: (i) to obtain conditions on $\Delta(\Gamma)$ so that, if Γ is large enough, eqn. (1.1) admits of a unique periodic solution; (ii) to determine an asymptotic expansion for this solution.

We give next a short qualitative description of the results and introduce the notation. We define:

$$x = y/\Gamma^{1/3} \qquad , \bar{t} = t - 3\pi/2 \tag{1.2}$$

so that eqn. (1.1) becomes:

$$\varepsilon \ddot{x} + 2\mu \dot{x} + x^3 = \sin \bar{t} \tag{1.3}$$

with

$$\varepsilon = 1/\Gamma^{2/3} \quad , \quad \mu = \Delta/\Gamma^{2/3} \tag{1.4}$$

and drop from now on the bar on t.

If, as $\varepsilon \to 0$, $\mu \to \mu_o$ = const $\neq 0$ (i.e. $\Delta \sim \Gamma^{2/3}$), eqn. (1.3) reduces in this limit to:

$$2\mu_o \dot{x} + x^3 = \sin t \tag{1.5}$$

which may be shown to admit of a unique periodic solution $x_{Po}(t)$ (cf. Sect. III). It is then easy to show that (1.1) admits of a (unique) periodic solution of the form:

$$x_P(t) = x_{Po}(t) + \varepsilon x_{P1}(t) + \ldots\ldots \tag{1.6}$$

and whose terms are obtained by iterating formally eqn. (1.4). However, if $\mu \longrightarrow 0$ as $\varepsilon \to 0$, eqn. (1.3) reduces to

$$x^3 = \sin t \tag{1.7}$$

with the solution

$$x_{oo}(t) = (\sin t)^{1/3} \tag{1.8}$$

Now, corrections to $x_{oo}(t)$ for small ε may no longer be obtained as before (in eqn. (1.6)), since the derivatives of $x_{oo}(t)$ at t = 0 are not finite. Thus, we expect a first change in the behaviour of the solutions of eqn. (1.3) as we cross the "asymptotic line" $\mu \sim 1$, i.e. $\Delta \sim \Gamma^{2/3}$.

If $\Delta \prec \Gamma^{2/3}$ (the symbols $\prec$, $\succ$, $\sim$ mean "asymptotically smaller, larger or equivalent to"), we attempt to obtain a boundary layer correction to $x_{oo}(t)$ near t = 0. Letting

$$t = \mu^{3/5}\tau \quad , \quad x = \mu^{1/5}\eta \tag{1.9}$$

eqn. (1.3) becomes:

$$\frac{\varepsilon}{\mu^{8/5}} \frac{d^2\eta}{d\tau^2} + 2\frac{d\eta}{d\tau} + \eta^3 = \frac{1}{\mu^{3/5}} \sin(\tau\mu^{3/5}) = \tau - \frac{\mu^{6/5}}{6}\tau^3 + \ldots. \tag{1.10}$$

We notice that, in a boundary layer of $0(\mu^{3/5})$, we may balance the restoring and the external force against the damping, as long as $\varepsilon \prec \mu^{8/5}$, i.e. $\Delta \succ \Gamma^{1/4}$.

If $\varepsilon \succ \mu^{8/5}$, the change of variables

$$t = \varepsilon^{3/8}\tau \quad , \quad x = \varepsilon^{1/8}\eta \tag{1.11}$$

leads to:

$$\frac{d^2\eta}{d\tau^2} + 2\frac{\mu}{\varepsilon^{5/8}}\frac{d\eta}{d\tau} + \eta^3 = \frac{1}{\varepsilon^{3/8}}\sin(\varepsilon^{3/8}\tau) \sim \tau - \varepsilon^{3/4}\frac{\tau^3}{6} + \ldots \quad (1.12)$$

in which, for small ε , the restoring and the external force are balanced against the acceleration. We expect thus another change in the form of the asymptotic expansion of the solutions of (1.3) across the "line" $\mu \sim \varepsilon^{5/8}$ ($\Delta \sim \Gamma^{1/4}$). Qualitatively, if $\Delta \succ \Gamma^{1/4}$, the motion in the boundary layer $O(\mu^{3/5})$ is strongly damped; if $\Delta \prec \Gamma^{1/4}$, however, we expect it to be oscillatory on a time scale of $O(\varepsilon^{3/8})$, near t = 0 and to be damped away in a time of $O(\varepsilon/\mu)$, which is asymptotically larger ($\varepsilon^{5/8} \succ \mu$).

If $\varepsilon \sim \mu$, i.e. $\Delta = O(1)$, asymptotic expansions for periodic solutions have been obtained in a recent paper by J. G. Byatt - Smith (Ref.[6]). Some of the formulae of Ref.[6] are relevant for the whole domain $\Delta \prec \Gamma^{1/4}$ and will appear below, however, obtained in a different (and independent) manner. The main features of the periodic solutions for high damping and forcing in the ranges of parameters described above have been already presented in Ref.[7]. This paper complements Ref.[7] with a more detailed analysis.

Concerning the problem of uniqueness, we prove that eqn. (1.1) admits of a unique periodic solution for Γ high enough, if , as $\Gamma \to \infty$,

$$\Delta(\Gamma)/\ln\Gamma \longrightarrow \infty \quad (1.13)$$

We also present arguments (but not a proof) that, if $\Delta(\Gamma) \sim \text{const}\ln\Gamma$, one of the Floquet multipliers related to our special periodic solution becomes larger than unity showing instability and thus the appearance of other, stable periodic solutions (Ref.[8]). With other words, the maxima of the domains in the Γ , Δ plane where several periodic solutions of (1.1) coexist (cf. Fig. 1 of Ref.[2]) are expected to increase logarithmically with Γ .

This latter expectation may be surprising and requires some comments. The stability of a periodic solution $x_p(t;\varepsilon)$ of (1.3) is read off the Floquet multipliers of the first variational equation to (1.3):

$$\varepsilon\ddot{x} + 2\mu\dot{x} + 3x_p^2(t;\varepsilon)\,x = 0 \quad (1.14)$$

Since, as $\varepsilon \to 0$, $x_p(t;\varepsilon)$ tends to $x_{oo}(t)$ in some way, we may presume that the

Floquet multipliers of (1.14) approach those of:

$$\varepsilon \ddot{x} + 2\mu \dot{x} + 3(\sin t)^{2/3} x = 0 \tag{1.15}$$

Concerning eqn. (1.15), we can prove the following:

<u>Lemma 1.1</u>: If ε is sufficiently small, the Floquet multipliers $\lambda_{1,2}(\varepsilon)$ of eqn. (1.15) are less than unity in absolute value if $\mu/\varepsilon \equiv \Delta > \Delta_o > 0$. If Δ is independent of Γ, the quantity $\max|\lambda_{1,2}(\varepsilon)|$ has maxima that are positioned for small ε (large Γ) equidistantly in $\Gamma^{1/3}$, with a spacing

$$\delta(\Gamma^{1/3}) = \frac{\pi}{\sqrt{3}\int_{-\pi/2}^{\pi/2}|\sin t'|^{1/3}\, dt'} \sim 1.403 \tag{1.16}$$

The proof of Lemma 1.1 is similar to that of Lemma 5.4, followed by the calculations of Sect. VI (eqns.(6.10),(6.13)). The proof allows one to evaluate Δ_o numerically: $\Delta_o \simeq 0.13$.

The fact that the Floquet multipliers of (1.14) may become larger than unity even if $\Delta \sim$ const $\ln \Gamma$, for large Γ (rather than $\Delta \sim \Delta_o$ as for (1.15)) is a consequence of the fact that $x_p(t;\varepsilon)$, although close in absolute value to $x_{oo}(t)$ uniformly on $[-\pi/2, \pi/2]$ oscillates (cf. eqn. (1.12)) in a time interval of $O(1/\Delta)$ with a local frequency that increases indefinitely with Γ and "resonates" there with the natural local frequency of (1.15). This behavior has similarities to that of the solutions of

$$\frac{d^2x}{d\tau^2} + \left(1 + \frac{\sin 2\tau}{\tau^{1/2}}\right) x = 0 \tag{1.17}$$

most of which are unbounded as $\tau \to \infty$, although the coefficient of x tends to unity as $\tau \to \infty$. (cf. Ref.[9], p. 136).

The asymptotic equidistant spacing of the maxima of the Floquet exponents mentioned in Lemma 1.1 subsists for eqn. (1.14), with the same value (1.16), if $\Delta \succ \ln \Gamma$ (see Sect.VI,VIII). This is a feature which has been observed to hold accurately for eqn. (1.1) even if $\Delta \sim$ const as $\Gamma \to \infty$ (Refs.[10,11]).

Finally, we comment on the methods of proof used in this paper. From eqn. (1.3) and its boundary layer versions, eqns. (1.10 - 12), we establish first a proposal $x_a(t)$ for a composite asymptotic expansion of the periodic solution, in the spirit of Refs.[12,13]; we prove then the validity of these expansions, i.e. that periodic solutions $x_p(t;\varepsilon)$ of (1.3) exist, that are close to them. This is done by Newton's method, with some resemblance to Ref.[14]. We require to this end estimates of the solutions of the variational equation to (1.3) at $x(t) = x_a(t)$. Controlled approximations to these solutions are offered by (suitably modified) WKB wave functions. They also allow the computation of the Floquet exponents of (1.14).

To prove uniqueness, we use a Liapunov function method to show first that all solutions $x(t)$ of (1.3) approach eventually $x_p(t)$, as close as we like as like as $\varepsilon \to 0$, if we stay away from the points $t = n\pi$. With the help of the solutions of the variational equation around $x_p(t)$ we write then an integral equation for $u(t) = x(t) - x_p(t)$ and show that its solution vanishes in fact as $t \longrightarrow \infty$, through all values of t, if ε is small enough. This approach is unduly complicated at least for large damping, where a Liapunov function method settles the problem of uniqueness completely[15]. The excuse for staying nevertheless with it is the uniformity of the approach over the whole range of parameters and the opportunity it gives to present at the same time the asymptotic expansions of the solutions.

The paper is organized as follows: Section II is preparatory: we describe the approach of the solutions of (1.3) to each other (in particular to a periodic solution), for time intervals excluding $t = n\pi$. We give then a general scheme of the uniqueness proofs. Sect. III disposes of the simple situation $1/\mu = O(1)$. Section IV presents the composite, i.e. inner and outer asymptotic expansions of the periodic solutions for $\Delta \succ \Gamma^{1/4}$ and of some special solutions of (1.3) if $\Delta \prec \Gamma^{1/4}$. We also place bounds on their residuals, i.e. on the extent to which they satisfy (1.3). Sect. V estimates the solutions of the variational equation to (1.3) around the approximate solutions $x_a(t)$ of the previous section. Sect. VI contains the proof of the existence and uniqueness of the periodic solutions for $\Delta \succ \Gamma^{1/4}$ and of some special, nonoscillatory ("outer left" and "outer right") solutions of (1.3) for $\Delta \prec \Gamma^{1/4}$. Section VII discusses a further, oscillatory solution of (1.3) for $t > 0$, with the help of which we propose a uniform approximation to a periodic solution of (1.3) for $\Delta \prec \Gamma^{1/4}$. Section VIII proves the existence and uniqueness of $x_p(t)$

for $\ln \ln \Gamma \prec \Delta \prec \Gamma^{1/4}$. Section IX contains some final remarks.

It is profitable to introduce right now some notation. If x is any twice continuously differentiable function on $[-\pi/2, \pi/2]$ (of class C^2), we call:

$$\mathcal{D}(x) \equiv \mathcal{D}_0(x) - \sin t \equiv \varepsilon \ddot{x} + 2\mu \dot{x} + x^3 - \sin t \tag{1.18}$$

the action of the "Duffing operator" on it. The residual of an approximant $x_a(t)$ to a solution $x(t)$ is $\mathcal{D}(x_a)$. $\mathcal{D}(\cdot)$ is a continuous operator from a Banach space $\mathbb{D}(-\pi/2, \pi/2)$ of C^2 odd periodic functions (i.e. $x(t) = -x(t+\pi)$), with the norm:

$$\|x\|_D = \varepsilon \sup_{|t|<\pi/2} |\ddot{x}| + 2\mu \sup_{|t|<\pi/2} |\dot{x}| + \sup_{|t|<\pi/2} |x| \tag{1.19}$$

to the space $\mathbb{C}(-\pi/2, \pi/2)$ of continuous odd periodic functions on $[-\pi/2, \pi/2]$ with the uniform norm:

$$\|y\|_C \equiv \|y\| = \sup_{|t|<\pi/2} |y(t)| \tag{1.20}$$

The Fréchet derivative of $\mathcal{D}(\cdot)$ at x acts on $\delta x \in \mathbb{D}(-\pi/2, \pi/2)$ through:

$$\mathcal{D}_x(x)(\delta x) = \varepsilon(\delta \ddot{x}) + 2\mu(\delta \dot{x}) + 3x^2(\delta x) \quad . \tag{1.21}$$

Its inverse, which leads from $\mathbb{C}$ back to $\mathbb{D}$ is given by the formula of the "variation of the constants":

$$k(t;f) \equiv [\mathcal{D}_x(x)]^{-1}(f)(t) = a_1(f)x_1(t) + a_2(f)x_2(t) + \int_{-\pi/2}^{t} \frac{1}{\varepsilon} \frac{x_1(t')x_2(t) - x_2(t')x_1(t)}{W(x_1,x_2)} \times f(t')\, dt' \tag{1.22}$$

where $x_1(t)$, $x_2(t)$ are two linearly independent solutions of $\mathcal{D}_x(x)(x_{1,2}) = 0$ (cf. (1.21)) and $a_1(f)$, $a_2(f)$ are chosen so that the result is in $\mathbb{D}$, i.e. $k(-\pi/2) = -k(\pi/2)$, $\dot{k}(-\pi/2) = -\dot{k}(\pi/2)$. If $x_i(t;f)$ denotes the integral in (1.22) and $\Delta x_k = x_k(-\pi/2) - x_k(\pi/2)$, k=1,2, it is true that:

$$a_k(f) = \frac{\Delta x_{3-k}\, \dot{x}_i(\pi/2;f) - \Delta \dot{x}_{3-k}\, x_i(\pi/2;f)}{\Delta \dot{x}_1\, \Delta x_2 - \Delta \dot{x}_2\, \Delta x_1} \quad , k=1,2 \tag{1.23}$$

In (1.22) and below:

$$W(x_1,x_2) \equiv x_1 \dot{x}_2 - x_2 \dot{x}_1 \tag{1.24}$$

is the Wronskian.

II. Preparation for the problem of uniqueness

Lemma 2.1: There exists a rectangle:

$$D : |x| < B_1 \quad , \quad \left|\frac{dx}{dt}\right| < \frac{B_2}{\sqrt{\varepsilon}} \tag{2.1}$$

so that all solution paths $(x(t), \dot{x}(t))$ of (1.3) eventually get inside it. The constants B_1, B_2 are independent of ε, μ if ε and ε/μ are sufficiently small.

Proof: We exhibit a Liapunov function $\Phi(p;x;t)$, with $p = dx/dt$ and depending on ε, with the following properties: (a) $\Phi(p;x;t) \longrightarrow 0$ as $|x|, |p| \rightarrow \infty$, uniformly in all directions and with respect to t, for all t; (b) $\Phi(p,x,t) > 0$ outside a rectangle:

$$D_1 : |x| < A_1 \quad , \quad \left|\frac{dx}{dt}\right| < \frac{A_2}{\sqrt{\varepsilon}} \tag{2.2}$$

(c) $d\Phi/dt\,(p;x;t) > \delta > 0$ outside (2.2) for all t. Then, one can show that there exists a rectangle (2.1), containing (2.2) in its interior, so that all solution paths eventually come into it (see Ref.[16], p.371, ch.VII,§3). The parameters of the rectangle D, eqn. (2.1), may be inferred from those of (2.2) as follows: let:

$$\Phi_{o1} = \min_t \min_{p,x \in \partial D_1} \Phi(p,x,t) \tag{2.3}$$

Then, choose D so that

$$\Phi_{o2} = \max_t \max_{p,x \in \partial D} \Phi(p,x,t) \tag{2.4}$$

obeys $\Phi_{o2} < \Phi_{o1}$.

Assume first $\varepsilon/\mu^2 < 1$; in this case, such a function Φ is offered in Ref.[16], p. 377, in an example due to G. E. H. Reuter (Ref.[17]):

$$\Phi(p;x;t) = \exp\left[-L(p;x;t)\right] \tag{2.5}$$

$$E(p;x;t) = \varepsilon\frac{p^2}{2} + \frac{x^4}{4} \tag{2.6}$$

$$D(p;x;t) = L(p;x;t) - E(p;x;t) \tag{2.7}$$

and

$$\begin{aligned} D(p;x;t) &= 0 && \text{if } p \geq \mu^{-1} \\ &= \frac{\varepsilon}{\mu}(p - \mu^{-1}) && \text{if } |p| \leq \mu^{-1},\ x \geq 2 \\ &= -2\frac{\varepsilon}{\mu^2} && \text{if } p \leq -\mu^{-1},\ x \geq 2 \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&= -(\varepsilon/\mu^2)x && \text{if } p \leq -\mu^{-1}, |x| < 2 \\
&= 2(\varepsilon/\mu^2) && \text{if } p < -\mu^{-1}, \; x < -2 \\
&= -\frac{\varepsilon}{\mu}(p - \mu^{-1}) && \text{if } |p| < \mu^{-1}, \; x < -2
\end{aligned}
$$

With this, we have in (2.2) $A_1 = 2$, $A_2 = \sqrt{\varepsilon}/\mu < 1$. Clearly, if $\varepsilon/\mu^2 \to \infty$, the quantity A_2 in (2.2) increases without bounds; we choose then instead of (2.6), (2.8):

$$E(p;x;t) = \varepsilon \frac{p^2}{2} + \frac{x^4}{4} - x \sin t \tag{2.9}$$

and

$$
\begin{aligned}
D(p;x;t) &= 0 && \text{if } p > \max\left[(|x|/\mu)^{1/2}, (A/\mu)^{1/2}\right] \qquad (2.10) \\
&= \varepsilon\,(p - \sqrt{x/\mu}) && \text{if } |p| < \sqrt{x/\mu}, \; x > A \\
&= -2\varepsilon\sqrt{x/\mu} && \text{if } p < -\sqrt{x/\mu}, \; x > A \\
&= -2\varepsilon\,(x/A)\sqrt{|x|/\mu} && \text{if } p < -\sqrt{A/\mu}, \; |x| < A \\
&= 2\varepsilon\sqrt{|x|/\mu} && \text{if } p < -\sqrt{|x|/\mu}, \; x < -A \\
&= -\varepsilon\,(p - \sqrt{|x|/\mu}) && \text{if } |p| < \sqrt{|x|/\mu}, \; x < -A
\end{aligned}
$$

Differentiation of (2.9-10) establishes the claim concerning $\Phi(p;x;t)$. A good choice is any $A > 3$. This latter choice of Φ satisfies then the conditions (a) - (c) above, provided $\mu < 1$, $\varepsilon/\mu < 1$. It is valid even if $\varepsilon \sim \mu^2$, which was marginal for (2.6 - 8). With (2.4), we may choose $B_1 \simeq 3$, $B_2 \simeq 9$. This ends the proof.

Remark: The ultimate boundedness of the solutions of (1.3) for fixed ε, μ is well known (see, e.g. Ref.[16],p.376). Lemma 2.1 asserts the independence of B_1, B_2 on ε if ε is small enough.

We show next that, if $\mu = o(1)$ and $\varepsilon/\mu = o(1)$, two solutions which enter and stay in the rectangle D of (2.1) approach each other as much as we wish, within a half period of the external force, provided only we allow ε to be appropriately small. For simplicity, we shall assume that, if $\mu < A$, eqn.(1.3) admits of a solution $x_o(t)$ which stays in D for $t > t_o$ and has the property:

(H1) There exist $a,b > 0$, so that $|x_o(t)| > a$, $|dx_o/dt| < b$, for $t \in [t_1, t_2]$, with $0 < t_1 < t_2 < \pi \pmod{\pi}$, $t > t_o$.

This can be proved directly, but we shall exhibit such a solution in the next sections (if $\mu = o(1)$, $x_o(t) \sim (\sin t)^{1/3}$, $0 < t < \pi$). Consider then another solution $x_1(t)$, which stays in D for $t > t_o$. We may then state:

Lemma 2.2: Assume $\mu / \varepsilon^{1/2} < A$ and $\varepsilon / \mu \longrightarrow 0$ as $\varepsilon \to 0$. Then, for ε sufficiently small, there exist constants k, C, independent of ε, so that, for $t \in [t_1, t_2]$:

$$\max\left[|x_1(t) - x_o(t)|, \sqrt{\varepsilon} \left| \frac{dx_1}{dt}(t) - \frac{dx_o}{dt}(t) \right| \right] < K \exp\left[- C \frac{\mu}{\sqrt{\varepsilon}} (t - t_1) \right] \tag{2.11}$$

Proof: The difference $u(t) \equiv x_1(t) - x_o(t)$ verifies:

$$\varepsilon \ddot{u} + 2 \mu \dot{u} + 3 x_o^2 u + 3 x_o u^2 + u^3 = 0 \tag{2.12}$$

Introduce:

$$v(t) = u(t) \exp\left[C \frac{\mu}{\sqrt{\varepsilon}} (t - t_1)\right] \tag{2.13}$$

which verifies

$$\varepsilon \ddot{v} + 2 \mu \dot{v} (1 - C) + v\left(3 x_o^2(t) + (C^2 - 2C)\frac{\mu^2}{\sqrt{\varepsilon}} \right) + 3 v^2 \exp\left[- C \frac{\mu}{\sqrt{\varepsilon}} (t - t_1)\right]$$
$$+ v^3 \exp\left[- 2C \frac{\mu}{\sqrt{\varepsilon}} (t - t_1)\right] = 0 \tag{2.14}$$

and consider the Liapunov function

$$L_v = \frac{(\varepsilon \dot{v} + 2 \mu \beta v)^2}{2} + \varepsilon G(v;t) \tag{2.15}$$

with $\beta = 1 - C$ and

$$G(v,t) = \int_o^v v F(v,t)\, dv \tag{2.16}$$

where $v F(v,t)$ denotes the last three terms of (2.14). The form $F(v,t)$ (and thus also $G(v,t)$) is positive definite for $t \in [t_1, t_2]$ if $C < a^2/(8A^2)$. Using (2.14), we obtain

$$\frac{dL_v}{dt} = - 2 \mu \beta \left(v^2 F(v,t) - \frac{\varepsilon}{2\mu\beta} \frac{\partial G}{\partial t} \right) = - 2 \mu \beta v^2 H(v,t) \tag{2.17}$$

It is easy to verify that if, e.g. $C < \min (1/2, a^2/8A^2)$ and ε/μ is small enough, then $H(v,t)$ is positive definite for $t \in [t_1, t_2]$. Thus the solution paths $(v, \dot{v})$ of (2.14) stay contained in the bounded domain:

$$L_v(t) < L_v(t_1) \tag{2.18}$$

for $t \in [t_1, t_2]$. But $L_v(t_1) = O(\varepsilon)$; therefore $G(v,t) = O(1)$ for $t \in [t_1, t_2]$ and, since $G(v,t) = v^2 F_1(v,t)$ with F_1 strictly positive definite, it follows that $v = O(1)$ for $t \in [t_1, t_2]$. Further, since $\varepsilon \dot{v} + 2\mu \beta v = O(\sqrt{\varepsilon})$, we obtain $\dot{v} = O(1/\sqrt{\varepsilon})$ for $t \in [t_1, t_2]$. Returning to (2.13), we obtain (2.11). This ends the proof.

Comment: According to (2.11), if $\Delta = \mu / \varepsilon = k \ln \Gamma = k_1 \ln 1/\varepsilon$, although $x_1(t)$ approaches $x_o(t)$ as much as we like within a half period, as $\varepsilon \to 0$, this may not be true for the derivatives if $k_1\pi < 1/2$. This indicates already that the line $\Delta \sim k \ln \Gamma$ may be a boundary for uniqueness questions.

For the range of paramaters $\mu \succ \varepsilon^{1/2}$, $\mu = O(1)$, we have:

Lemma 2.3 : If $\mu < A$, $\mu^2/\varepsilon > B$, A/B sufficiently small, there exist constants K, C, independent of ε, so that, if $t \in [t_1+\delta, t_2]$:

$$\max_t \left(|x_1(t) - x_o(t)| , \mu |\dot{x}_1(t) - \dot{x}_o(t)| \right) < K \exp\left[- \frac{C}{\mu} (t - t_1)\right] \qquad (2.19)$$

The quantity δ tends to zero as A/B approaches zero.

Proof: This runs similarly to that of Lemma 2.2; however, we let now:

$$v(t) = u(t) \exp\left[C(t - t_o)/\mu\right] \qquad (2.20)$$

As before, we prove that dL_v/dt is negative definite for $t \in [t_1, t_2]$, if A/B and C are appropriately small. Now, $L_v(t_1) = O(\mu^2)$. It follows that :

$(\beta = 1 - C\varepsilon/\mu^2)$

$$\varepsilon \frac{dv}{dt} + 2 \mu \beta v = g(t) = O(\mu) \qquad (2.21)$$

Eqn. (2.21) has the solution:

$$v(t) = D \exp\left[-2\mu\beta(t-t_1)/\varepsilon\right] + \varepsilon^{-1} \int_{t_1}^{t} g(t') \exp\left[- 2\mu \beta(t-t')/\varepsilon\right] dt'$$

$$= O(1) \qquad (2.22)$$

Now, the quantity dg/dt may be obtained from the differential equation satisfied by v(t). In view of the estimate (2.22) for v(t), $t \in [t_1,t_2]$, it follows that $dg/dt = O(1)$. Integrating (2.22) by parts and computing dv/dt we obtain:

$$\dot{v} = O(\mu^{-1}, \varepsilon^{-1} \exp(- 2\mu\beta(t-t_1)/\varepsilon)) \qquad (2.23)$$

In (2.23) and the following, $O(x,y) = O(\max(x,y))$. If ε/μ is sufficiently small (i.e. if A/B is small), the second term in (2.23) becomes smaller that $1/\mu$ in a time $\delta < t_2 - t_1$. Clearly, $\delta \to 0$ as $\varepsilon \to 0$. It follows that $\dot{v} = O(1/\mu)$ in $[t_1+\delta, t_2]$. Returning to (2.20), we obtain (2.19). This ends the proof.

These Lemmas are used to prove uniqueness in conjunction with a quantitative version of a well known stability theorem (Ref.[9],p.86) for sets of equations with periodic coefficients. To state it for our situation, assume the solution of reference $x_o(t)$ above is 2π (odd-) periodic. Thus $x_o(t) = x_p(t)$. Then, the following holds:

Theorem 2.1: (Ref.[9]). Assume that all solutions of:

$$\varepsilon \ddot{v} + 2 \mu \dot{v} + 3 x_p^2 v = 0 \tag{2.24}$$

tend to zero as $t \to \infty$. Then the solution $u(t)$ of (2.12) with initial conditions $u(t_o)$, $\dot{u}(t_o)$ tends to zero as $t \to \infty$, provided $u(t_o), \dot{u}(t_o)$ are sufficiently small.

The general procedure to establish uniqueness is: (a) determine the size of a domain around the origin in the $(u(t_o), u(t_o))$ plane $(t_o \neq n\pi)$, so that all solutions of (2.12) starting in it tend to zero as $t \to \infty$; (b) use Lemmas 2.1-3 to show that every solution of (2.12) reaches at some t_o the interior of this domain. The manner to solve problem (a), i.e. the quantitative part of Theorem 2.1, follows the same pattern over the whole range of parameters considered and we describe it below.

First, we notice that the periodicity of $x_p(t)$ allows us to settle question (a) using only the values of $v_{1,2}(t)$ over a finite time interval. With this, sufficient conditions for $u(t)$ to tend to zero as $t \to \infty$ are that: (i) the Poincaré mapping $(u(t_o),\dot{u}(t_o)) \longrightarrow (u(t_o+n\pi),\dot{u}(t_o+n\pi))$ given by the solutions of (2.12) maps, for some choice of t_o, n and all sufficiently small M, neighbourhoods:

$$\mathcal{U}(M) \ : \ \| u,\dot{u} \|_2 \equiv |u| + g(\varepsilon)|\dot{u}| < M \tag{2.25}$$

of $(u = 0, \dot{u} = 0)$ into $\mathcal{U}(kM)$, with $k < 1$ and that (ii) the solutions $u(t),\dot{u}(t)$, with initial conditions in $\mathcal{U}(M)$ stay bounded on $[t_o, t_o + n\pi]$ by a quantity that vanishes as $M \to 0$. The task is to determine how small M must be, so that conditions (i), (ii) are fulfilled.

To estimate $u(T)$, $\dot{u}(T)$, $(T = t_o + n\pi)$, we use two solutions $v_1(t;\varepsilon), v_2(t;\varepsilon)$ of (2.24) and the initial conditions $u(t_o)$, $\dot{u}(t_o)$ to turn (2.12) into an integral equation for $u(t)$ on $[t_o,T]$. Explicitly:

$$u(t) = b_1 v_1(t;\varepsilon) + b_2 v_2(t;\varepsilon) + [\varepsilon W(v_1,v_2)(t_o)]^{-1} \int_{t_o}^{t} \exp[\tfrac{2\mu}{\varepsilon}(t'-t_o)] \times$$

$$\times [v_1(t')v_2(t) - v_1(t)v_2(t')][-3x_o(t')u^2 - u^3]\, dt' \tag{2.26}$$

where b_1, b_2 are chosen so that the initial conditions are fulfilled. We obtain $\dot{u}(t)$ in terms of $u(t)$ by differentiating (2.26). We regard the two equations for $u(t)$, $\dot{u}(t)$ as a mapping $B(u,\dot{u})$ from a space $\mathcal{C}_\lambda$ of (pairs of) functions continuous on $[t_o,T]$ with the norm:

$$\|(u,\dot u)\|_\lambda = \sup_{t_o<t<T} \exp[\lambda(t-t_o)]\|u(t),\dot u(t)\|_2 \qquad (2.27)$$

into itself. The quantity λ is chosen depending on the range of the parameters Δ, Γ, but in general such that $\lambda < \min(\lambda_1,\lambda_2)$, where $\tilde\mu_{1,2} \equiv \exp[-\lambda_{1,2}]$ are the Floquet multipliers (not necessarily distinct) associated to (2.24); $\lambda_{1,2} > 0$ by the hypothesis of Theorem 2.1.

We formulate now conditions for the mapping $B(u,\dot u)$ to be contractive in a ball of radius M_o in $\mathbb{C}_\lambda$. Let to this end $K_B(\varepsilon;\lambda)$ be an upper bound to:

$$\|K\| = \sup_{t_o\leq t\leq T} \int_{t_o}^{t} \exp[(2\tfrac{\mu}{\varepsilon} - 2\lambda)(t'-t_o)]\exp[\lambda(t-t_o)]\left[|v_1(t')\, v_2(t)| + |v_1(t)v_2(t')|\right]dt' \qquad (2.28)$$

and $K_D(\varepsilon;\lambda)$ a bound for a similar quantity with $v_1(t)$, $v_2(t)$ replaced by their derivatives. Then, a sufficient condition for the ball $\|u,\dot u\|_\lambda < M_o$ to be mapped into itself is:

$$\sum_{i=1}^{2} |b_i|\|v_i,\dot v_i\|_\lambda + 4(\varepsilon W(t_o))^{-1}\|K_B(\varepsilon;\lambda),K_D(\varepsilon;\lambda)\|_2\, M_o^2 < M_o \qquad (2.29)$$

This is fulfilled if:

$$\sum_{i=1}^{2} |b_i|\|v_i,\dot v_i\|_\lambda = M_o/2 \qquad (2.30)$$

and:

$$M_o < \min\left[1, \frac{W(t_o)}{8\|K_B,K_D\|_2}\right] \equiv S(\varepsilon) \qquad (2.31)$$

Because of the choice (2.30), eqn. (2.31) is a condition on the $|b_i|$, eqn. (2.26) and thus on $u(t_o)$, $\dot u(t_o)$. It is easy to verify that eqn.(2.31) is also sufficient for contraction. Since:

$$|b_i| = \frac{|W(v_{3-i},u)|}{|W(v_1,v_2)|}(t_o) < \frac{|u(t_o)|\|\dot v_{3-i}\|_\lambda + |\dot u(t_o)|\|v_{3-i}\|_\lambda}{|W(v_1,v_2)|(t_o)} \qquad (2.32)$$

where $\|x\|_\lambda = \sup[|x(t)|\exp\lambda(t-t_o)]$, we may write a stronger version of (2.31) by using for M_o eqn. (2.30) and for $|b_i|$ the right hand side of (2.32). Using also the inequality $\|(u,\dot u)\|_\lambda < \|u\|_\lambda + g\|\dot u\|_\lambda$, we obtain the condition:

$$R(|u(t_o)|, |\dot u(t_o)|) < S(\varepsilon) \qquad (2.33)$$

where:

$$R(|u(t_o)|,|\dot u(t_o)|) = W(t_o)^{-1}\left[(2gP_D + \tilde M)|u(t_o)| + (2P + g\tilde M)|\dot u(t_o)|\right] \qquad (2.34)$$

with $P_D = \|\dot v_1\|_\lambda\|\dot v_2\|$, $P = \|v_1\|_\lambda\|v_2\|_\lambda$, $\tilde M = \|v_1\|_\lambda\|\dot v_2\|_\lambda + \|\dot v_1\|_\lambda\|v_2\|_\lambda$.

To summarize, if we choose the size M of $\mathcal{U}(M)$ in (2.25) so small that the maximum value $R(M)$ of $R(|u(t_o)|, |\dot{u}(t_o)|)$ on it satisfies (2.33), then the solutions of (2.12) starting in $\mathcal{U}(M)$ obey on $[t_o, T]$:

$$\| u(t),\dot{u}(t)\|_2 < R(M)\exp[-\lambda(t-t_o)] \tag{2.35}$$

Now, the value $R(M)$ decreases obviously to zero with M, so that condition (ii) for $u(t)$ to vanish as $t \to \infty$ is fulfilled. We still have to verify that, for such values of M, $\mathcal{U}(M)$ is mapped at $t = T$ into $\mathcal{U}(kM)$, with $k < 1$. This follows in turn from the inequality $R(|u|,|\dot{u}|) < k_R \|u,\dot{u}\|_2$, for some $k_R(\varepsilon) > 0$, provided we show that λ may be chosen so that, apart from (2.33), also $k_R \exp[-\lambda(T-t_o)] < 1$. If $\mu < A$, A small, it is convenient to choose in (2.25) $g = (P/P_D)^{1/2}$, so that:

$$R(|u|,|\dot{u}|) = \left[(\|v_1\|_\lambda \|\dot{v}_2\|_\lambda)^{1/2} + (\|\dot{v}_1\|_\lambda \|v_2\|_\lambda)^{1/2}\right]^2 W^{-1} \|u,\dot{u}\|_2 \equiv T(\varepsilon)\|u,\dot{u}\|_2 \tag{2.36}$$

and λ must be chosen so that $T(\varepsilon)\exp[-\lambda(T-t_o)] < 1$. In the course of the paper, we give, in various ranges of parameters, estimates of the quantities used above and show that the appropriate choices of the parameter λ can indeed be made. This closes the preparation for the uniqueness problem.

Controlled approximants to solutions $v_1(t;\varepsilon), v_2(t;\varepsilon)$ of (2.24) on finite intervals are obtained in Sects. III,V,VIII; they are also used to obtain the Floquet multipliers associated to $x_p(t)$. To this end, we recall the Floquet matrix, which is such that:

$$v_i(t+\pi) = f_{i1}\, v_1(t) + f_{i2}\, v_2(t) \tag{2.37}$$

has matrix elements given by: $(i,j = 1,2)$

$$f_{ij} = (-1)^{j+1} \frac{\dot{v}_{3-j}(-\pi/2)\, v_i(\pi/2) - v_{3-j}(-\pi/2)\, \dot{v}_i(\pi/2)}{W(v_1,v_2)(t = -\pi/2)} \tag{2.38}$$

The quantities $v_i(\pi/2)$, $\dot{v}_i(\pi/2)$ contain a factor $\exp[-(\mu\pi/\varepsilon)]$, so that we may write:

$$F = \exp\left(-\frac{\mu}{\varepsilon}\pi\right)\tilde{F} \tag{2.39}$$

and $\det \tilde{F} = 1$. The Floquet multipliers are given by:

$$\tilde{\mu}_{1,2} \equiv \exp\left(-\lambda_{1,2}\pi\right) = \exp\left(-\frac{\mu}{\varepsilon}\pi\right)\left\{\frac{1}{2}\operatorname{Tr}\tilde{F} \pm \left[\left(\frac{1}{2}\operatorname{Tr}\tilde{F}\right)^2 - 1\right]^{1/2}\right\} \tag{2.40}$$

III. The situation at very high damping: $\Gamma^{2/3}/\Delta = 1/\mu = O(1)$

This situation lends itself to an easy analysis; the approach used in this Section will be imitated later for other ranges of parameters. It is convenient to change variables to $z = \mu x$:

$$\chi \ddot{z} + 2\dot{z} + \sigma z^3 = \sin t \tag{3.1}$$

with $\varepsilon/\mu = \chi$, $\sigma = 1/\mu^3$. Clearly, $\chi \to 0$ as $\varepsilon \to 0$.

We obtain immediately an asymptotic expansion of a 2π periodic solution $z_p(t)$ of (3.1): let z_p^o be the 2π periodic solution (pending a proof of its existence and uniqueness) of:

$$2\dot{z} + \sigma z^3 = \sin t \tag{3.2}$$

Then:

$$z_p \sim z_p^o + \chi z_p^{(1)} + \chi^2 z_p^{(2)} + \ldots.. \tag{3.3}$$

with $z_p^{(1)}$ the (unique) periodic solution of:

$$2\dot{z} + 3\sigma z_p^{(o)^2} z = -\ddot{z}_p^{(o)} \tag{3.4}$$

etc. The existence and uniqueness of $z_p^{(1)}$, $z_p^{(2)}$,... is obvious due to the linearity of the corresponding equation. The existence of $z_p^{(o)}$ is also clear: the time 2π Poincaré map of (3.2) transforms an interval $[-k,k]$ with $k\sigma^{1/3} > 1$ into itself and has thus at least one fixed point; further, the Liapunov function

$$L(y_1,y_2) = (z_1 - z_2)^2 \tag{3.5}$$

obeys $dL/dt < -\sigma k^4/8$, if $|z_1 - z_2| > k$, so that $z_p^{(o)}$ is unique (see Ref.[16], p. 379, ch. VII, §3). Thus, (3.3) is well defined. Also, the uniqueness of $z_p^{(i)}$, $i = 0,1,2,\ldots$ means that they share the symmetry of the corresponding equations, i.e. they are odd periodic.

Let now:

$$z_n(t) = \sum_{k=0}^{n} \chi^k z_p^{(k)}(t) \tag{3.6}$$

Clearly, with (1.18), the residual of (3.6) is ($\tilde{\mathcal{D}}$ refers to the version (3.1) of (1.3)):

$$\tilde{\mathcal{D}}(z_n) = O(\sigma\chi^{n+1}) \tag{3.7}$$

We prove next the existence of a solution of (3.1) with the asymptotic expansion (3.6) using Newton's method. With the notation of the Introduction (and the obvious changes due to the replacement of (1.3) with (3.1)), Newton's method means obtaining the correction $\Delta z \equiv z_n(t) - z_p(t)$ to (3.6) through iteration of the nonlinear equation in $\mathbb{D}(-\pi/2,\pi/2)$:

$$\Delta z = [\tilde{\mathcal{Z}}_x(z_n)]^{-1}(\tilde{\mathcal{Z}}(z_n)) + [\tilde{\mathcal{Z}}_x(z_n)]^{-1}\ \tilde{\mathcal{U}}(z_n;\Delta z) \equiv \mathcal{A}(z_n;\Delta z) \tag{3.8}$$

with $\tilde{\mathcal{Z}}_x^{-1}$ of (1.22) and $\tilde{\mathcal{U}}(z_n;\Delta z)$ the mapping from $\tilde{\mathbb{D}}(-\pi/2,\pi/2)$ to $\mathbb{C}(-\pi/2,\pi/2)$ given by:

$$\tilde{\mathcal{U}}(z_n;\Delta z) = 3\sigma\ z_n\ (\Delta z)^2 + \sigma\ (\Delta z)^3 \tag{3.9}$$

The sequence of iterates converges if $\mathcal{A}(\Delta z)$ maps some ball $\|\Delta z\| < M$ in $\tilde{\mathbb{D}}(-\pi/2,\pi/2)$ into itself and is contractive there; the error $\|\Delta z\|$ of z_n is given by the smallest value of M with these properties.

Let $\|[\tilde{\mathcal{Z}}_x(z_n)]^{-1}\| = K$. If $\|\Delta z\| < M$, it is true that:

$$\|\mathcal{A}(\Delta z)\|_D < K\left[\|\tilde{\mathcal{Z}}(z_n)\|_{\mathbb{C}} + O(\sigma M^2)\right] \tag{3.10}$$

and

$$\|\mathcal{A}(\Delta z_1) - \mathcal{A}(\Delta z_2)\| < k\,\sigma\,M\,\|\Delta z_1 - \Delta z_2\|_D \tag{3.11}$$

Thus, to settle the question of convergence, we have to bound K, i.e. estimate the magnitude of the solutions of (1.21), in our case of:

$$\chi\ddot{z} + 2\dot{z} + 3\sigma z_n^2 z = 0 \tag{3.12}$$

Concerning this, we have:

<u>Lemma 3.1</u>: Eqn. (3.12) has two linearly independent solutions of the form:

$$z_{(1)}(t) = g_1(t)\ \exp\left[-\frac{1}{\chi}\int_{-\pi/2}^{t} d_1(t')\,dt'\right] \tag{3.13}$$

$$z_{(2)}(t) = g_2(t)\ \exp\left[-\sigma\int_{-\pi/2}^{t} d_2(t')\,dt'\right] \tag{3.14}$$

where

$$d_1(t) = 1 + (1 - 3\sigma\chi z_n^2)^{1/2} > 1 \tag{3.15}$$

$$d_2(t) = \frac{1}{\chi\sigma}\left(1 - (1 - 3\sigma\chi z_n^2)^{1/2}\right) \sim 3z_n^2/2 \tag{3.16}$$

and

$$g_{1,2}(t) = \left[1 - 3\sigma\chi z_n^2\right]^{-1/4}\ h_{1,2}(t) \tag{3.17}$$

with $h_{1,2}(t) = 1 + O(\sigma\chi^2)$ and with first and second derivatives of $O(\sigma\chi^2)$, for $t \in [-\pi/2, \pi/2]$.

<u>Proof</u>: Without the factors $h_{1,2}(t)$, eqns. (3.13-17) are WKB approximants to the solutions of (3.12). By means of standard changes of dependent and independent variables (see Refs.[18-21]):

$$\xi = \int_{-\pi/2}^{t} \left[\frac{1}{\chi} - 3\,\sigma\, z_n^2(t')\right]^{1/2} dt' \equiv \int_{-\pi/2}^{t} \phi(t')^{1/2} dt' \tag{3.18}$$

$$w = \exp\left[(t + \pi/2)/\chi\right]\left[1 - 3\chi\sigma z_n^2\right]^{1/4} z \tag{3.19}$$

the equation for w may be written as:

$$\chi \frac{d^2 w}{d\xi^2} - (1 + \chi R(\phi))\, w = 0 \tag{3.20}$$

with:

$$R(\phi) = \frac{5}{16}\phi^{-3}\left(\frac{d\phi}{dt}\right)^2 - \frac{1}{4}\phi^{-2}\frac{d^2\phi}{dt^2} \tag{3.21}$$

Transforming (3.20) to an integral equation, one sees that the corrections $h_{1,2}(t)$ satisfy:

$$h_1(t) = 1 + \chi^{1/2}\int_{-\pi/2}^{t}\left[1 - e^{-2(\xi-\xi')/\sqrt{\chi}}\right] h_1(t')\, R(\phi)\, \phi^{1/2}(t')\, dt' \tag{3.22}$$

and a similar equation for $h_2(t)$ with $-\pi/2$ replaced by $\pi/2$. Gronwall's Lemma yields then the estimates in the statement of the Lemma. This ends the proof.

<u>Comment</u>: The manipulations above are standard. They have been written only for further reference. The estimates rest on the fact that z_n, $\dot z_n$, $\ddot z_n$ are all O(1).

With (1.22), (1.23) and (1.19) one estimates easily, using (3.13-14):

$$W(z_{(1)}, z_{(2)})(t) = O(\chi^{-1}) \exp\left[-\frac{2}{\chi}(t + \pi/2)\right] \tag{3.23}$$

and

$$K = \left\| \left[\breve{\mathcal{D}}_x(z_n)\right]^{-1} \right\| = O(1) \tag{3.24}$$

From (3.10) and using (3.7), we see that, if we choose $M = O(\sigma\chi^n)$, it is true that $\|\mathcal{A}(\Delta z)\| < M$, if $\|\Delta z\| < M$; since, in (3.11), it is clear that $k\sigma M < 1$ (for small χ), we have established the existence of a periodic solution $z_p(t)$, with the expansion (3.6) and an error of $O(\sigma\chi^n)$. Since the integer n in (3.6) is at our disposal, we can approximate $z_p(t)$ and its first two derivatives by means of $z_n(t)$ as well as we wish. In particular, z_p, $\dot z_p$, $\ddot z_p$ are all O(1) and the estimates of Lemma 3.1 stay unchanged even if we replace z_n by z_p.

The (absolute values of the) Floquet exponents are:

$$\lambda_1 = \frac{1}{\chi\pi}\int_{-\pi/2}^{\pi/2} d_1(t')\, dt' \quad , \qquad \lambda_2 = \frac{\sigma}{\pi}\int_{-\pi/2}^{\pi/2} d_2(t')\, dt' \tag{3.25}$$

with d_1, d_2 of (3.15-16) and z_n replaced by z_p.

With the help of (3.13-14), we evaluate next the various quantities appearing in Sect. II, eqns. (2.25-36), needed to prove the uniqueness of $z_p(t)$. There are the (obvious) changes of notation $\varepsilon \to \chi$, $\mu \to 1$ and the nonlinearity in (2.26) contains a factor σ. From eqn. (3.22), we notice that $|h_{1,2}(t) - 1| = O(\chi)$ on time intervals $T - t_o \sim 1/\chi\sigma$, and so is $\dot{h}_{1,2}(t)$ $(h_1(t_o) = 1, \dot{h}_1(t_o) = 0, h_2(T) = 1, \dot{h}_2(T) = 0$ at $T = t_o + 1/\chi\sigma$). With this, we choose in (2.25) $g(\varepsilon) = \chi$ and let $\lambda = k\sigma$, $0 < k < \lambda_2/\sigma$. With (3.23), $\chi W(t_o) = \text{const} \neq 0$. Further, in eqn. (2.28), using the remark above on $h_{1,2}(t)$ and the fact that $(1/\chi)\, d_1(t) + \sigma\, d_2(t) = 2\hbar/\varepsilon$, we obtain:

$$\|K\| < \text{const} \sup_{t_o < t < T} \int_{t_o}^{t} e^{-\lambda(t'-t_o)} \left[e^{\int_{t'}^{t}(\lambda - d_1(t'')/\chi)dt''} + e^{\int_{t'}^{t}(\lambda - \sigma d_2(t''))dt''} \right] dt$$

$$< \text{const} \int_{t_o}^{T} e^{-k\sigma(t'-t_o)} = C_1/\sigma \equiv K_B(\varepsilon;\lambda) \tag{3.26}$$

In deriving (3.26), we have bounded the second term in the brackets by a constant, for any t', t, in view of the choice of $\lambda < \lambda_2$. Similarly, $K_D(\varepsilon;\lambda) = C_2$ so that, recalling the factor σ in the nonlinearity,

$$S(\varepsilon) = \min\left[1, \frac{C_3}{8(C_1 + \chi\sigma C_2)}\right] \tag{3.27}$$

with C_3 a constant (independent of ε). On the other hand, using (3.13-14) and the choice of λ:

$$R(|u(t_o)|, |\dot{u}(t_o)|) = |u(t_o)|(1 + 3\chi\sigma) + |\dot{u}(t_o)|(3 + \varepsilon\chi) \tag{3.28}$$

The maximum of $R(|u(t_o)|, |\dot{u}(t_o)|)$ on $\mathcal{U}(M)$ is less than 4M (for small ε) so that, if $M < S(\varepsilon)/4$, all trajectories starting in $\mathcal{U}(M)$ are bounded by $4M\times$ $\exp[-\lambda(T-t_o)]$ on $[t_o, T]$. In particular, for ε small enough:

$$\|u(T), \dot{u}(T)\|_2 < 4\,\|u(t_o), \dot{u}(t_o)\|_2\, e^{-k\sigma(1/\chi\sigma)} < k_1 \|u(t_o), \dot{u}(t_o)\|_2 \tag{3.29}$$

with $0 < k_1 < 1$, which shows that, for such values of M, $\mathcal{U}(M)$ is mapped into $\mathcal{U}(k_1 M)$ at $t = t_o + T$. Thus all trajectories starting at t_o in $\mathcal{U}(M)$ vanish as $t \to \infty$. We still have to verify that all trajectories of (3.12) reach $\mathcal{U}(M)$ at some t_o, with $M < S(\varepsilon)/4$. Since $|\dot{u}(t_o)| = O(\hbar/\sqrt{\varepsilon})$, $\|u, \dot{u}\|_2 < |u(t_o)| + O(\sqrt{\varepsilon})$ and we only have to prove that any solution that enters D, eqn. (2.1), comes at some time t_o within a distance from the origin less than, say, $S(\varepsilon)/5$. A much stronger statement is, in fact, true:

<u>Lemma 3.2</u>: Let $u(t) = z(t) - z_p(t)$, with $z(t)$ a solution of (3.1); there exists a time t_o, so that $|u(t_o)| = O(\chi)$.

<u>Proof</u>: Consider the Liapunov function (cf. Lemmas 2.2-3):

$$L = \frac{(\chi\dot{u} + 2u)^2}{2} + \chi\sigma G(u,t) \tag{3.30}$$

where:

$$G(u,t) = 3\, z_p^2\, u^2/2 + z_p\, u^3 + u^4/4 \tag{3.31}$$

One verifies that:

$$\frac{dL}{dt} = -2\sigma\, u^2\left[\frac{3}{2}\, z_p(2z_p - \chi\dot{z}_p) + (3z_p - \frac{\chi\dot{z}_p}{2})\, u + u^2\right] = -2\sigma u^2 Q(\chi;u;t) \tag{3.32}$$

The discriminant of Q is:

$$\Delta(\chi;t) = -3(z_p^2 - \chi\, z_p\, z_p - \chi^2\dot{z}_p^2/4\,) \tag{3.33}$$

It is negative, except for those times when the ratio $z_p/\dot{z}_p$ lies in an interval of width $O(\chi)$, centered at a point with abscissa $O(\chi)$. Now, $|\dot{z}_p| = O(1)$, so that, at these times, $|z_p| = O(\chi)$ itself. From (3.32), we verify then that $dL/dt < 0$ outside a stripe $|u| = O(\chi)$ in the $(u,\dot{u})$ plane. Moreover, given a constant C, we can find a stripe S_1, containing the previous one and also of width $O(\chi)$, so that, outside it:

$$\frac{dL}{dt} < -C\chi^4 < 0 \tag{3.34}$$

With this, the remaining part of the proof is a slight variation of the one in Ref.[16], p. 374, Lemma 2. We know that, for every solution path of (3.1), there exists a time t_1 so that, for $t > t_1$, it stays contained in the rectangle D of Lemma 2.1. Consider then the function: $(t > t_1)$

$$\bar{L} = L\, \exp[C_1 \chi^4 t] \tag{3.35}$$

with $0 < C_1 < C$: it is such that $\bar{L} \to \infty$ in $D \cap \complement S_1$, as $t \to$, but $d\bar{L}/dt < 0$ there. This cannot be reconciled with the hypothesis that the solution path never leaves $D \cap \complement S_1$. This ends the proof.

Summing up, we have thus proved:

Theorem 2.1: Eqn. (1.3) admits of a unique periodic solution $x_p(t) \doteq z_p(t)/\mu$ if ε is sufficiently small and $\mu > A > 0$; $z_p(t)$ has the asymptotic expansion (3.3).

IV. The situation $\mu = o(1)$. Outer and inner expansions.

If t lies sufficiently far away from $n\pi$, we may iterate (1.3) formally, i.e. we expect a solution exists with the asymptotic expansion (the "outer expansion"):

$$x_o(t) = \sum_{k,l \geq 0} \mu^k \varepsilon^l \; x_{kl}(t) \tag{4.1}$$

where

$$x_{oo}(t) = (\sin t)^{1/3} \tag{4.2}$$

$$x_{1o}(t) = -2\,\dot{x}_{oo}(t)/(3\,x_{oo}^2\,), \quad \text{etc.} \tag{4.3}$$

In general, we may state:

Lemma 4.1 :

$$x_{kl}(t) = t^{1/3 - 5k/3 - 8l/3} \sum_{q=0}^{\infty} a_{klq}\, t^{2q} \tag{4.4}$$

where the sum is uniformly and absolutely convergent on $[-\pi+\sigma, \pi-\sigma]$, for any $\sigma > 0$.

The proof is straightforward and performed by induction with respect to, say, l; at each fixed l, the statement is verified by induction with respect to k.

We write in the following:

$$x_o^{(K,L)}(t) = \sum_{k,l \leq K,L} \mu^k \varepsilon^l \; x_{kl}(t) \tag{4.5}$$

If the damping and the forcing are large, but $\mu < A$, with A sufficiently small, we expect eqn. (4.5) to be even an approximant to a periodic solution of (1.3), for $t \neq n\pi$: on one hand, the corrections to $(\sin t)^{1/3}$ are odd periodic and small and on the other, the disturbances at the passage through $t = n\pi$ are damped away in a very short time after which we recover (4.5).

If $A\Gamma^{2/3} > \Delta > B\Gamma^{1/4}$ $(1 \succ \mu \succ \varepsilon^{5/8})$, for sufficiently small A and sufficiently large B, an approximate solution to the boundary layer equation (1.10) may be attempted as a suitably truncated formal expansion in both small parameters $\mu^{6/5}$, $\nu \equiv \varepsilon/\mu^{8/5}$ (the inner expansion):

$$x_{in}(\tau) \sim \mu^{1/5} \sum_{q,l} \mu^{6q/5} \nu^l \; \eta_{ql}(\tau) \equiv \mu^{1/5} \sum_{l} \nu^l \tilde{\eta}_l(\tau) \equiv \mu^{1/5} \sum_{q} \mu^{6q/5} \bar{\eta}_q(\tau) \tag{4.6}$$

The $\eta_{kl}(\tau)$ are solutions of the equations:

$$2\,\frac{d\eta_{oo}}{d\tau} + \eta_{oo}^3 = \tau \tag{4.7a}$$

$$2\,\frac{d\eta_{o1}}{d\tau} + 3\,\eta_{oo}^2\,\eta_{o1} = -\frac{d^2\eta_{oo}}{d\tau^2} \tag{4.7b}$$

$$\vdots$$

$$2\,\frac{d\eta_{1o}}{d\tau} + 3\,\eta_{oo}^2\,\eta_{1o} = -\tau^3/6 \tag{4.8a}$$

$$2\,\frac{d\eta_{11}}{d\tau} + 3\,\eta_{oo}^2\,\eta_{11} = -\frac{d^2\eta_{10}}{d\tau^2} - 6\,\eta_{oo}\,\eta_{o1}\,\eta_{1o}\,, \text{ etc.} \tag{4.8b}$$

Since we expect the periodic solution of (1.3) to be close to $(\sin t)^{1/3}$ for small $t < 0$, but $\tau \to -\infty$, it is natural to choose the $\eta_{qo}(\tau)$ as those solutions of (4.7), (4.9) which behave as $\tau \to -\infty$ like $\tau^{1/3}$, $\tau^{7/3}$,etc. The behaviour of the $\eta_{ql}(\tau)$, $l > 0$, should be that given by the power of τ on the right hand side, divided by $\tau^{2/3}$ $(\sim \eta_{oo}^2)$ and ignoring the derivative term. For definiteness, we may state:

<u>Lemma 4.2</u> : Expansion (4.6) is well defined.

<u>Proof:</u> For eqn. (4.7a), the existence of a solution behaving like $\tau^{1/3}$ as $\tau \to -\infty$ follows by changing variables to $v = \eta/\tau^{1/3} - 1$, $\sigma = \tau^{5/3}$ so that it assumes the form:

$$\frac{dv}{d\sigma} + \frac{9}{5} v \left(1 + v + \frac{v^2}{3} \right) + \frac{v+1}{5\sigma} = 0 \tag{4.9}$$

with the boundary condition $v \to 0$ as $\sigma \to -\infty$. Using the variation of the parameters, (4.9) may be transformed into an integral equation, which can be shown to have a unique solution in a ball $\sup\limits_{\sigma < -\sigma_o} |v(\sigma)| < M$, for a sufficiently large σ_o. Uniqueness is easily obtained: the difference of two solutions of (4.7a) obeys an homogeneous equation with exponentially increasing solutions as $\tau \to -\infty$.

All other equations, apart from (4.7a), are linear and the sought solution is produced by the method of variation of parameters. The equations have to be solved recursively, for increasing q,l. This ends the proof.

<u>Comment</u> : In eqn. (4.6), the $\tilde{\eta}_l(\tau)$, $\bar{\eta}_q(\tau)$ are in turn the solutions of:

$$2 \frac{d\tilde{\eta}_o}{d\tau} + \tilde{\eta}_o^3 = \mu^{-3/5} \sin(\mu^{3/5}\tau) \text{ , etc.} \tag{4.10}$$

or

$$\nu \frac{d^2\bar{\eta}_o}{d\tau^2} + 2\frac{d\bar{\eta}_o}{d\tau} + \bar{\eta}_o^3 = \tau \quad \text{, etc.} \tag{4.11}$$

with the boundary conditions of periodicity on an interval of length $2\pi/\mu^{3/5}$ or, in turn, that $\bar{\eta}_o \sim \tau^{1/3}$ as $\tau \to -\infty$ (for (4.11)). For eqn. (4.10), the existence and uniqueness of the $\tilde{\eta}_i$ follows from the same argument as in Sect. III (surrounding eqns. (3.4-5)). Ref.[22] proves the same for the $\bar{\eta}_i$ in eqn.(4.11). We refer to the $\tilde{\eta}_l$, $\bar{\eta}_q$ expansions below.

We have next:

<u>Lemma 4.3</u>: The asymptotic expansion of $\eta_{ql}(\tau)$ as $\tau \to -\infty$ is:

$$\eta_{ql}(\tau) \sim \tau^{2q + 1/3 - 8l/3} \sum_k a_{klq} \tau^{-5k/3} \tag{4.12}$$

where a_{klq} are the <u>same</u> constants as in (4.4).

<u>Proof</u>:(i) First we prove that:

$$\eta_{ql}(\tau) \sim \tau^{2q + 1/3 - 8l/3}\left[\sum_k b_{klq}\tau^{-5k/3} + O(\tau^{-5(k+1)/3})\right] \tag{4.13}$$

with the b_{klq} real constants. They are determined by inserting (4.12) in the eqn. no. (q,l) of the set (4.7 - 8) and equating to zero the coefficients of the various powers of τ. We write then:

$$\eta_{ql} = \eta_{ql,k_o} + u \tag{4.14}$$

with η_{ql,k_o} the sum in (4.13) truncated after k_o terms. Substitution in the (q,l) eqn. of (4.7-8) yields a linear equation for u (nonlinear if q=l=0), with a right hand side of $O(\tau^{2q+1-8l/3-5(k+1)/3})$; if k_o is sufficiently large, depending on q,l, the boundary condition is $u \to 0$ as $\tau \to -\infty$. This allows either an explicit solution of the equation for u or a contraction proof (if q=l=0) that a solution exists with the required falloff at infinity.

(ii) The equations for the b_{koo} may be written implicitly :

$$2\sum_k b_{koo}(1/3 - 5k/3)\,\tau^{-5(k+1)/3} + \left(\sum_k b_{koo}\tau^{-5k/3}\right)^3 = 1 \tag{4.15}$$

and may be solved recurrently, for increasing k. There is no difficulty to verify that the b_{klo} may be expressed in terms of the $b_{k'l'o}$, either with $l' < l$ and $k' \leq k$ or $l' = l$ and $k' < k$. A similar statement is true for the b_{klq}, $q > 0$.

We argue now that the coefficients a_{klq} of (4.4) are obtained by solving precisely the same equations. Since the solution of the latter is unique, we shall have established the identity $b_{klq} = a_{klq}$.

To show this, we notice that the expression $x_o^{(K,L)}(t)$, eqn. (4.5), when sub - stituted in (1.3) and use is made of (4.4), yields a meromorphic function of $\theta = t^{1/3}$, the coefficients of the Laurent expansion around $\theta = 0$ are polynomials in ε, μ; the coefficients of the latter vanish identically, up to those of ε^{L+1}, μ^{K+1}. We may try to determine the a_{klq} from this (infinite) set of equations.

An equation in this set is indexed by the power s of θ which we consider, and a pair of indices k,l for the powers of ε, μ. Since:

$$s/3 = 2q + 1/3 - 5k/3 - 8l/3 \tag{4.16}$$

we may change indices to q,k,l. Now, the equations with q=0, l=0 form a closed set, i.e. no other values of q,l are involved. They may be written as:

$$2\sum_{k\in K_o} a_{koo}\ \mu^{k+1}(1/3 - 5k/3)\ t^{-5(k+1)/3} + \left(\sum a_{koo}\ \mu^{k}\ t^{-5/3}\right)^3 = 1 \qquad (4.17)$$

Using (1.9), we verify that (4.17) is transformed into (4.15). With care, but no special difficulties, one verifies that the same is true for the higher values of 1,q. This ends the (short) exposition of the proof of Lemma 4.3.

Concerning the behaviour as $\tau \to +\infty$, we have:

Lemma 4.4: As $\tau \to +\infty$, $\eta_{q1}(\tau)$ have the same asymptotic behaviour (4.12), with the same a_{klq}.

Proof: We show that, in fact, all solutions of the equation indexed with q,1 in the set (4.7-8) have the same asymptotic behaviour (4.12), as $\tau \to +\infty$. As in the proof of Lemma 4.3, use of (4.13-14) in the corresponding equation of (4.7-8) leads to equations for the b_{klq} and for the function $u(\tau)$. As before, we show that $b_{klq} = a_{klq}$ and consider the equation for $u(\tau)$ only in the situation q=1=0 (the others are simpler). Using $\eta(\tau) \equiv \eta_{oo,k_o}(\tau)$ and changing variables to:

$$u(\tau) = w(\tau)\ \exp\left[-\ 3C\int_{\tau_o}^{\tau}\eta^2(\tau')d\tau'\right] \qquad (4.18)$$

we obtain an equation for w, for which we consider the Liapunov function $L(w)=w^2$. It is true that:

$$\frac{dL}{d\tau} < L^{1/2}\ \ R(\tau)\ \exp\left[3C\int_{\tau_o}^{\tau}\eta^2(\tau')d\tau'\right] \qquad (4.19)$$

with $R(\tau)$ the right hand side of the equation for u, of $O(\tau^{1-5(k+1)/3})$. Integrating (4.19) and returning to u, we obtain

$$|u(\tau)| < |u(\tau_o)|\exp\left[-3C\int_{\tau_o}^{\tau}\eta^2(\tau')d\tau'\right] + \int_{\tau_o}^{\tau}R(\tau')\ \exp\left[-3C\int_{\tau'}^{\tau}\eta^2(\tau'')d\tau''\right]d\tau' \qquad (4.20)$$

The last term in (4.20) behaves like $R(\tau)$ and this ends the proof.

With this, we write down proposals for uniform approximations to odd periodic solutions of (1.3) as follows (δ is a small positive number): if $A_1\Gamma^{2/3-\delta} < \Delta < A\Gamma^{2/3}$, we take:

$$x_a(t) = \sum_{1=0}^{L} \nu^1\ \tilde{\eta}_1(t/\mu^{3/5};\mu) = x_i^{(L)}(t) \qquad (4.21a)$$

with the $\tilde{\eta}_1$ of eqn. (4.10); if $B_1\Gamma^{1/4+\delta} < \Delta < A_1\Gamma^{2/3-\delta}$:

$$x_a(t) = \chi_o(t;\mu^{\alpha})\ x_o^{(K,L)}(t) + \chi_i(t;\mu^{\alpha})\ x_i^{(Q,L)}(t) \qquad (4.21b)$$

where $\chi_i(t;\mu^{\alpha})$ is unity if $|t| < a\mu^{\alpha}$, zero if $|t| > b\mu^{\alpha}$, $(b>a)$ and is of class C^2; α obeys $0<\alpha<3/5$; also, throughout $(-\pi/2, \pi/2)$

$$\chi_o(t;\mu^{\alpha}) + \chi_i(t;\mu^{\alpha}) \equiv 1 \qquad (4.22)$$

In (4.21), $x_i^{(Q,L)}$ is a (Q,L) truncation of (4.13), analogous to (4.5). Finally, if $A_1 \Gamma^{1/4+\delta} > \Delta > B\Gamma^{1/4}$, we write:

$$x_a(t) = \chi_o(t;\mu^\alpha)\, x_o^{(K,L)}(t) + \chi_i(t;\mu^\alpha)\, \bar{x}_i^{(Q)}(t) \tag{4.21c}$$

where $\bar{x}_i^{(Q)}(t)$ is a truncation of the expansion (4.6) in terms of the $\bar{\eta}_q(\tau)$, eqn. (4.11).

With Lemmas 4.3, 4.4, we can evaluate the residual of $x_a(t)$, eqn. (4.21). For simplicity, we let $P = \min(K,L,Q)$ and consider the situation of P large. More detailed estimates than those below are available in Refs.[22,23]. We have:

Lemma 4.5: If $A\Gamma^{2/3} > \Delta > B\Gamma^{1/4}$, there exist constants c_1, $c_2 > 0$ (depending on the choice of δ in the domains assigned to (4.21)), so that

$$\sup_{|t|<\pi/2} |\mathcal{D}(x_a)(t)| < c_1\, \varepsilon^{c_2 P} \tag{4.23}$$

with $x_a(t)$ of (4.21).

Proof: If $A\Gamma^{2/3} > \Delta > A_1\Gamma^{2/3-\delta}$, substitution of (4.21a) into (1.3) leads, in complete analogy to (3.7) to $\mathcal{D}(x_a) = O(\mu^{3/5}\ \nu^{1+1})$; since, if $\delta < 5/12$, $\nu = O(\varepsilon^p)$, with $p = p(\delta) > 0$, the estimate (4.23) holds. If $A_1\Gamma^{2/3-\delta} > \Delta > B_1\Gamma^{1/4+\delta}$, we use (4.21b) and

$$\mathcal{D}(x_a) \equiv \chi_i(\mathcal{D}_o(x_i) - \sin t) + \chi_o(\mathcal{D}_o(x_o) - \sin t) + 2\,(d\chi_i/dt)\left[\varepsilon(\dot{x}_i - \dot{x}_o) + \mu(x_i - x_o)\right] + (d^2\chi_i/dt^2)\,\varepsilon\,(x_i - x_o) + \chi_i\,\chi_o\,(x_i - x_o)^2\, T(x_i;x_o) \equiv$$

$$T_i + T_{out} + T_m \tag{4.24}$$

where T_m is concentrated on $[a,b]\mu^\alpha$, $0<\alpha<3/5$ and is proportional to the quality of the matching of x_{in} with x_{out}, and of their derivatives. Now, for $|t|>a\mu^\alpha$, $T_{out} = O(\mu^k/t^{5k/3})$, whereas for $|t|<b\mu^\alpha$, $T_{in} = O(\mu^{6Q/5}\ \tau^{2Q+1}, \nu^L)$. If $\delta > 0$, then $\mu = O(\varepsilon^{q(\delta)})$, $\nu = O(\varepsilon^{p(\delta)})$, with $q(\delta), p(\delta) > 0$, so that T_{in}, T_{out} are $O(\varepsilon^{c(\alpha,\delta)P})$, for some $c(\alpha,\delta) > 0$. Irrespective of the value of α, if $\delta \to 0$, then $c(\alpha,\delta) \to 0$, because either μ or ν become of order unity. Thus, to obtain (4.23), we have to keep $\delta > 0$. An estimate of the same order is true for T_m. Now, at fixed δ, if $\alpha \to 0$, the terms $\eta_{qo}\mu^{6q/5}$ of the inner expansion become comparable in magnitude and ~ 1, so that $c(\alpha,\delta) \to 0$. The terms of the outer expansion acquire order unity as $\alpha \to 3/5$, and $c(\alpha,\delta) \to 0$ again. One may find an optimal value of α, for which $c(\alpha,\delta)$ has a maximum, but we do not need this explicitly below (see Refs.[22,23]). If $B_1\Gamma^{1/4+\delta} > \Delta$

$> B\Gamma^{1/4}$, use of (4.21c) allows us to drop the term $O(\nu^{L})$ in the estimate of T_{in} and we recover (4.23). This ends the discussion of Lemma 4.5.

If $\Delta < B\Gamma^{1/4}$, the appropriate inner equation is (1.12); if $\Delta/\Gamma^{1/4} \to 0$ as $\Gamma \to \infty$, the small parameters are $\varepsilon^{3/8} = \Gamma^{-1/4}$ and $\mu/\varepsilon^{5/8} \equiv \gamma \equiv \nu^{-5/8} \equiv \Delta/\Gamma^{1/4}$. The inner expansion relevant for our problem reads, with the variables (1.11):

$$x_i(t) = \varepsilon^{1/8}\,\eta_i(t) = \varepsilon^{1/8}\sum_q \varepsilon^{3q/4}\,\eta_q(\tau;\gamma) \tag{4.25}$$

where the $\eta_q(\tau;\gamma)$ are those solutions behaving in turn like $\tau^{2q+1/3}$ as $\tau \to -\infty$ of the equations:

$$\frac{d^2\eta_o}{d\tau^2} + 2\gamma\,\frac{d\eta_o}{d\tau} + \eta_o^3 = \tau \tag{4.26}$$

$$\frac{d^2\eta_1}{d\tau^2} + 2\gamma\,\frac{d\eta_1}{d\tau} + 3\eta_o^2\,\eta_1 = -\tau^3/6 \quad , \text{ etc.} \tag{4.27}$$

Without proof (which is similar to that of Lemmas 3.2, 3.3), we state:

<u>Lemma 4.6</u>: Expansion (4.25) is well defined. The asymptotic expansion of the $\eta_q(\tau)$ as $\tau \to -\infty$ is given by:

$$\eta_q(\tau) \sim \tau^{2q+1/3}\sum_{k,1} a_{klq}\,\tau^{-5k/3-8l/3}\,\gamma^k \tag{4.28}$$

where a_{klq} are the same constants as in (4.4).

The behaviour of the $\eta_q(\tau)$ as $\tau \to +\infty$ is, however, more complicated, if $\gamma \to 0$ as $\varepsilon \to 0$: the reason is that a new time scale appears $\tau \sim 1/\gamma$; in general, for $\tau > 0$, the solutions $\eta_q(\tau)$ have oscillations, which are damped out in a time $t \sim \varepsilon/\mu$ $(\tau \sim 1/\gamma)$; this becomes arbitrarily large on the reduced time scale τ , as $\varepsilon \to 0$. We discuss this in detail in Sect. VII. Thus, in analogy to Lemma 4.4, we can only state:

<u>Lemma 4.7</u>: If $C\Gamma^{1/4} < \Delta < B\Gamma^{1/4}$, for some $C > 0$, the asymptotic behaviour of the $\eta_q(\tau)$ as $\tau \to +\infty$ is the same as (4.28), with the same constants.

<u>Comment</u> : To be sure, the asymptotic expansion of $\eta_q(\tau)$ as $\tau \to +\infty$ is the same for all values of the parameter γ , i.e. even without the restriction $\Delta/\Gamma^{1/4} > C$. It is not uniform with respect to γ (or ε), however: i.e. given a constant C_o, large enough, the inequality:

$$|\eta_q(\tau) - \eta_{qk_ol_o}(\tau)| < C_o \max\left[\gamma^{k_o+1}\,\tau^{-5(k_o+1)/3}, \tau^{-8(l_o+1)/3}\right]\tau^{2q+1/3} \tag{4.29}$$

where $\eta_{qk_o l_o}$ is a (k_o,l_o) truncation of (4.28) holds for $\tau > \tau_o(\gamma)$ and $\tau_o(\gamma) \to \infty$ as $\gamma \to 0$.

Despite of the restrictions on the range of Δ in Lemma 4.7, we still need expansions (4.4),(4.28) to obtain approximants to special solutions of Duffing's eqn. (1.3), even if $\Delta \prec \Gamma^{1/4}$. If $C\Gamma^{1/4} < \Delta < B\Gamma^{1/4}$, we can write, as in (4.12), using (4.4), (4.28) and an obvious notation:

$$x_a(t) = \chi_o(t;\varepsilon^\alpha)\, x_o^{(K,L)}(t) + \chi_i(t;\varepsilon^\alpha)\, x_i^{(Q)}(t) \tag{4.30}$$

as a uniform approximant on $|t| < \pi/2$ to an odd periodic solution of (1.3). In (4.30), α obeys $0 < \alpha < 3/8$. If $\Delta < C\Gamma^{1/4}$, we write only for $t < 0$, and the same interval for α :

$$x_{La}(t) = \chi_{oL}(t;\varepsilon^\alpha)\, x_o^{(K,L)}(t) + \chi_i(t;\varepsilon^\alpha)\, x_i^{(Q)}(t) \tag{4.31}$$

where $\chi_{oL}(t;\varepsilon^\alpha)$ is the restriction of $\chi_o(t;\varepsilon^\alpha)$ to $t < 0$. Concerning (4.30), (4.31), we can state, in analogy to Lemma 4.5:

<u>Lemma 4.8</u> : The residuals $\mathcal{D}(x_a)$, $\mathcal{D}(x_{La})$ of (4.30), (4.31) are such that:

$$\sup \left| \mathcal{D}(x_{a,La})(t) \right| < c_o\, \varepsilon^{c_1 P} \tag{4.32}$$

with $P = \min(Q,K,L)$, for some c_o, $c_1 > 0$. (For x_{La}, the supremum is taken on $-\pi/2 < t < 0$.

We refer to Ref.[23] for more detailed estimates.

As already announced, the continuation to $t > 0$ of the solution of (1.3) which is approximated for $t < 0$ by (4.31) has oscillations around a reference level given approximately by $(\sin t)^{1/3}$. We make next this reference level more precise, by exhibiting an approximant to it, analogous to (4.31). Clearly, the outer expansion has the same form for $t < 0$ and $t > 0$. The difficulty is that the boundary condition $\eta_q(\tau) \sim \tau^{2q+1/3}$ as $\tau \to +\infty$ does not select a unique solution of (4.26-27), since, in fact, all solutions behave like this. (cf. Comment following Lemma 4.7). However, whereas most solutions reach this behaviour for $\tau \succ 1/\gamma$,there are some which approach it more rapidly. We turn now to the description of this special class.

Consider first eqn. (4.26)($q = 0$). We look for a solution of it in the form:

$$\eta_{oR}(\tau) = \sum_{k=0}^{r} \eta_{ok}\gamma^k + \gamma^{r+1} v(t) \equiv \eta_o^{(r)}(t) + u(t) \tag{4.33}$$

where

$$\frac{d^2\eta_{oo}}{d\tau^2} + \eta_{oo}^3 = \tau \tag{4.34}$$

and

$$\frac{d^2\eta_{ok}}{d\tau^2} + 3\,\eta_{oo}^2\,\eta_{ok} = -\,f(\eta_{oo},\ldots,\eta_{o,k-1},d\eta_{oo}/d\tau,\ldots d\eta_{o,k-1}/d\tau) \tag{4.35}$$

(k=1,2...). In (4.35), f is a polynomial combination of its arguments (of degree at most three). The η_{ok} are those solutions of (4.34-35) which behave like $\tau^{1/3 - 5k/3}$ as $\tau \to +\infty$. One verifies easily that, since the damping term is absent, this boundary condition selects unique solutions for (4.34-35) (and that such solutions exist). Further, v(t) obeys:

$$\frac{d^2v}{d\tau^2} + 2\gamma\frac{dv}{d\tau} + 3\eta_o^{(r)^2}v + 3\,\eta_o^{(r)}\gamma^{r+1}\,v^2 + \gamma^{2r+2}\,v^3 = k(\tau) \tag{4.36}$$

where $k(\tau)$ is a function behaving like $\tau^{1/3 - 5(r+1)/3}$ as $\tau \to +\infty$. It is easy to show that (4.36) admits of bounded solutions for $\tau > \tau_o$, γ small and any such solution may be used in (4.33).

Now, if we set up an equation for u(t), eqn. (4.33), we can prove that for $\tau \succ \gamma^{-1}$, all solutions obey:

$$|u(\tau)| < \text{const}\,|k(\tau)| \tag{4.37}$$

Since the asymptotic expansion of the η_{ok} is (cf. (4.28)):

$$\eta_{ok} \sim \sum_{l} a_{klo}\,\tau^{1/3 - 5k/3 - 8l/3} \tag{4.38}$$

it is true that, for $\tau > \tau_o$

$$\left|\eta_{oR} - \sum_{k=0}\gamma^k\,\eta_{ok,l_o}\right| < \text{const}\,\max\left(\gamma^{r+1},\,\tau^{1/3 - 5(l_o+1)/3}\right) \tag{4.39}$$

where η_{ok,l_o} is an l_o truncation of (4.38). In view of (4.37), if $\tau \succ 1/\gamma$, we may even replace the right hand side of (4.39) by that of (4.28) with q = 0.

In a strictly similar manner, we derive solutions $\eta_{qR}(\tau)$ of (4.27) and its analogues. We write then an inner expansion:

$$x_{iR}^{(Q)}(t) = \varepsilon^{1/8}\sum_{q=1}^{Q}\eta_{qR}\,\varepsilon^{3q/4} \tag{4.40}$$

and an approximant to a solution of (1.3) for $t > 0$ ($\Delta < C\,\Gamma^{1/4}$) as:

$$x_{aR}(t) = \chi_i(t;\varepsilon^\alpha)\,x_{iR}^{(Q)}(t) + \chi_{oR}(t;\varepsilon^\alpha)\,x_o^{(K,L)}(t) \tag{4.41}$$

($\chi_{oR} + \chi_{oL} = \chi_o$, cf. eqn. (4.31)). If the power α, defining the interval $[a,b]\varepsilon^\alpha$ where the matching of $x_{iR}^{(Q)}(t)$ to $x_o(t)$ is performed, is such that

$\varepsilon^{\alpha-3/8} > 1/\gamma$, the $\eta_{qR}(\tau)$ are given by (4.33) and the residual of $x_{aR}(t)$ is the same as for $x_{aL}(t)$ (cf. (4.32)). If $\mu \sim \varepsilon^{\beta}$, the inequality for α is $\alpha+\beta < 1$; if $\alpha+\beta > 1$, which is the case for small damping, there is an additional term $O(\gamma^{r+1})$ in the estimate of the difference $x_i - x_o$ (cf. eqn.(4.39)). However, in this range, $\gamma \sim \varepsilon^{\delta}$, for some $\delta > 0$, so that we may state:

Lemma 4.9: Let $P = \min(Q,K,L)$, $S = \min(Q,K,L,r)$. Then:

$$\sup \left| \mathcal{D}(x_{aR})(t) \right| = O(\varepsilon^{c_1 P(S)}) \tag{4.42}$$

for some $c_1 > 0$; the estimate in brackets is valid if $\alpha+\beta > 1$.

This closes the discussion of the inner and outer expansions.

V. The variational equation.

In this Section, we consider the variational equation associated to the approximate solutions $x_a(t)$ to eqn. (1.3), presented in the previous Section:

$$\varepsilon \ddot{x} + 2\mu \dot{x} + 3 x_a^2 x = 0 \tag{5.1}$$

Letting:

$$x = w \exp\left[-\frac{\mu}{\varepsilon}(t+\pi/2)\right] \tag{5.2}$$

we obtain:

$$\varepsilon \ddot{w} + (3 x_a^2(t) - \frac{\mu^2}{\varepsilon}) w \equiv \varepsilon \ddot{w} + \phi(t;\varepsilon) w = 0 \tag{5.3}$$

which is the standard WKB form. If $\phi(t) > 0$, i.e. if $\mu^2/\varepsilon > 3$, ($A\Gamma^{2/3} > \Delta > \sqrt{3}\,\Gamma^{1/3}$), the solutions of (5.1) are obtained in complete analogy to Lemma 3.1:

Lemma 5.1: If $A\Gamma^{2/3} > \Delta > \sqrt{3}\,\Gamma^{1/3}$, eqn. (5.1) admits of two linearly independent solutions of the form:

$$x_1(t) = g_1(t) \exp\left[-\frac{\mu}{\varepsilon}\int_{-\pi/2}^{t} e_1(t')\,dt'\right] \tag{5.4}$$

$$x_2(t) = g_2(t) \exp\left[-\frac{1}{\mu}\int_{-\pi/2}^{t} e_2(t')\,dt'\right] \tag{5.5}$$

where

$$e_1(t) = 1 + (1 - 3\eta x_a^2)^{1/2}, \qquad e_2(t) = \left[1 - (1 - 3\eta x_a^2)^{1/2}\right]/\eta \tag{5.6}$$

with $\eta = \varepsilon/\mu^2$ and

$$g_{1,2}(t) = [1 - 3\,\eta\, x_a^2(t)]^{1/4}\ h_{1,2}(t) \tag{5.7}$$

The functions $h_{1,2}(t)$ are bounded on $[-\pi/2, \pi/2]$, with bounded derivatives, independently of ε.

The proof is identical to that of Lemma 3.1.

If $\Delta < \sqrt{3}\,\Gamma^{1/3}$, eqn. (5.3) has two turning points $t_{L,R}$, situated at the roots of (see Ref.[22] for details):

$$3\,(\sin t_{L,R})^{2/3} = \mu^2/\varepsilon \tag{5.8}$$

i.e. $|t_{L,R}| \simeq (1/3^{3/2})\,\mu^3/\varepsilon^{3/2}$. The wavelength near $t = 0$ is $O(\varepsilon^{1/2}/(\mu/\varepsilon^{1/2}))$ $= O(\varepsilon/\mu)$. The latter is much less than $|t_R - t_L|$ if $\varepsilon/\mu \prec \mu^3/\varepsilon^{3/2}$, i.e. $\mu \succ \varepsilon^{5/8}$ or $\Delta \succ \Gamma^{1/4}$. In quantum mechanics language, the solutions of (5.1) for $\sqrt{3}\,\Gamma^{1/3} > \Delta > B\,\Gamma^{1/4}$ with a sufficiently large B are the wave functions for the penetration of a "thick" barrier. The wave functions are expressed more easily if we change variables to:

$$\theta = t\lambda \equiv t\ \varepsilon^{3/2}/\mu^3 \tag{5.9}$$

so that the turning points $\theta_{L,R}$ lie at a fixed distance apart (i.e. for small ε, independent of the relation $\mu = \mu(\varepsilon)$). Eqn. (5.3) changes to:

$$\rho^2 \frac{d^2 w}{d\theta^2} + (\,Q(\varepsilon;\mu;\theta) - 1\,)\, w = 0 \tag{5.10}$$

with ($\rho \to 0$ as $\varepsilon \to 0$)

$$\rho^2 = \varepsilon^5/\mu^8 \qquad , \qquad Q = 3(\varepsilon/\mu^2)\, x_a^2(t) \tag{5.11}$$

With this, we have: (Refs.[19-21])

Lemma 5.2: Eqn. (5.10) admits of two linearly independent solutions, $w_1(\theta)$, $w_2(\theta)$, which are uniformly approximated (as described below) on $[-\pi\lambda/2, \pi\lambda/2]$ by: $(c,d > 0)$

$$\begin{aligned} \bar{w}_1(\theta) &= \frac{1}{s}\, r(\theta)\ \cos\Big[\frac{1}{\rho}\,\Psi(\theta_R,\theta) + \frac{\pi}{4}\Big] && \theta > \theta_R + d \\ &= r(\theta)\ \exp\Big[-\frac{1}{\rho}\,\Psi(\theta_L,\theta)\Big] && \theta_L + c < \theta < \theta_R - d \\ &= r(\theta)\ \sin\Big[\frac{1}{\rho}\,\Psi(\theta;\theta_L) + \frac{\pi}{4}\Big] && \theta < \theta_L - c \end{aligned} \tag{5.12}$$

$$\begin{aligned} \bar{w}_2(\theta) &= s\, r(\theta)\ \sin\Big[\frac{1}{\rho}\,\Psi(\theta_R,\theta) + \frac{\pi}{4}\Big] && \theta > \theta_R + d \\ &= r(\theta)\ \exp\Big[\frac{1}{\rho}\,\Psi(\theta_L,\theta)\Big] && \theta_L + c < \theta < \theta_R - d \\ &= r(\theta)\ \cos\Big[\frac{1}{\rho}\,\Psi(\theta;\theta_L) + \frac{\pi}{4}\Big] && \theta < \theta_L - c \end{aligned} \tag{5.13}$$

with

$$\Psi(a,b) = \int_a^b |Q - 1|^{1/2}\, d\theta \quad , \quad r(\theta) = |Q(\theta) - 1|^{-1/4} \tag{5.14}$$

and

$$s = \exp\left[\frac{1}{\rho} \int_{\theta_L}^{\theta_R} (1 - Q)^{1/2}\, d\theta' \right] \tag{5.15}$$

is the barrier transmission factor. On $[\theta_L - c, \theta_L + c]$, $w_1(\theta)$ is uniformly approximated by:

$$\overline{w}_1(\theta) = \left[\frac{3}{2\rho}\, \Psi_0(\theta_L;\theta)\right]^{1/6} r_0(\theta - \theta_L) \ \ \mathrm{Ai}\ \frac{3}{2} \quad (\ \ _L; \ \ )^{2/3} \tag{5.16}$$

similar estimates holds near $\ _R$ with Ai(x) replaced by Bi(x) and for $w_2(\ \ )$ in the reverse direction (Ai(x), Bi(x) are the Airy functions of the first and second kind).[24] For $\theta > \theta_R + d$, the solution $w_1(\theta)$ which obeys $w_1(\pi\lambda/2) = \overline{w}_1(\pi\lambda/2)$, $dw_1/d\theta\ (\pi\lambda/2) = d\overline{w}_1/d\theta\ (\pi\lambda/2)$ is approximated absolutely to $O(\rho/s)$ on $[\theta_R - C\rho^{2/3}, \pi\lambda/2]$; if $\theta \in (\theta_L + C\rho^{2/3}, \theta_R - C\rho^{2/3})$, the approxima - tion is relative to $O(\rho)$; for $\theta < \theta_L + C\rho^{2/3}$, it is absolute and $O(\rho\ )$. The first and second derivatives are approximated as above, e.g. for $\theta < 0$, to $O(1)$ and $O(1/\rho)$, in turn. Similar statements hold for $w_2(\theta)$, which obeys the same initial conditions as $\overline{w}_2(\theta)$ at $\theta = -\pi\lambda/2$.

The proof of this statement is contained in Refs.[19-21]; the error estimates are applications of Gronwall's Lemma, in the manner of Lemma 3.1. Lemma 5.2 gives more information than is needed for qualitative statements of existence and uniqueness. We use it in evaluating the Floquet multipliers associated to the periodic solution. We only notice the increase like $\rho^{-1/6}$ of the wave functions near the turning points (cf. eqn. (5.16)).

If $\Delta < B\Gamma^{1/4}$, then we can no longer use the WKB approximation between the turning points; changing variables as in (1.11) in eqn. (5.3), we obtain (notation of eqn. (4.25)):

$$\frac{d^2 w}{d\tau^2} + (3\, \eta_i^2(\tau) - \zeta^2)\, w = 0 \tag{5.17}$$

which we cannot solve even approximately. However, we can still state:

<u>Lemma 5.3</u> : If $\Delta < B\Gamma^{1/4}$, eqn. (5.3) admits on $[-\pi/2, 0]$ of two linearly independent solutions of the form:

$$w_1(t) = \chi_{oL}(t;C_\epsilon)\, u_{1o}(t;\varepsilon) + \chi_i(t;C_\varepsilon)\, u_{1i}(\tau;\varepsilon) + O(\varepsilon^m) \tag{5.18a}$$

$$w_2(t) = \chi_{oL}(t;C_\varepsilon)\, u_{2o}(t;\varepsilon) + \chi_i(t;C_\varepsilon)\, u_{2i}(t;\varepsilon) + O(\varepsilon^m) \tag{5.18b}$$

where u_{1o}, u_{2o} are WKB approximants to solutions of (5.3) for $t < -b\,C_\varepsilon$ and u_{1i}, u_{2i} are solutions of eqn. (5.17), chosen so as to match u_{1o}, u_{2o} in turn on $-aC_\varepsilon < t < -bC_\varepsilon$. The solutions $u_{1i}(\tau;\varepsilon)$, $u_{2i}(\tau;\varepsilon)$ are uniformly bounded, on $-aC_\varepsilon < t < 0$ and tend uniformly (together with their derivatives) to a limit as $\varepsilon \to 0$ for τ on $[-\varepsilon^{-\delta},0]$, for a sufficiently small δ. In (5.18), m is a positive number, $m > 1/16$, which depends only on C_ε, if P (or S), eqn. (4.42), are large enough. The quantity C_ε has the property $C_\varepsilon \to 0$ and $\varepsilon^{3/8}/C_\varepsilon \to 0$ as $\varepsilon \to 0$ (we may take $C_\varepsilon \sim \varepsilon^{\alpha}$, as in the construction of x_R, x_L, in Sect. IV, but need not to, which is of use in Section VIII).

We describe first the matching and choice of the solutions; it depends on a parameter $\tau_o > 0$. We choose:

$$\begin{pmatrix} u_{1o} \\ u_{2o} \end{pmatrix}(t;\varepsilon) = 3^{1/4}\,\frac{\varepsilon^{1/16}}{\phi^{1/4}}\begin{pmatrix} \cos \\ -\sin \end{pmatrix}\left[\int_t^{-bC_\varepsilon} \frac{\phi(t')^{1/2}}{\sqrt{\varepsilon}}\,dt' + \psi_L(\varepsilon;\tau_o)\right] \tag{5.19}$$

with $\phi(t;\varepsilon)$ of eqn. (5.3); $\psi_L(\varepsilon;\tau_o)$ is defined with the help of (cf.eqn. (5.17)):

$$\phi_{in}(\tau;\varepsilon) = 3[\eta_i^0(\tau)]^2 - \lambda^2 \; ; \tag{5.20}$$

it is given by:

$$\psi_L(\varepsilon;\tau_o) = \int_{-bC_\varepsilon}^{-\tau_o\varepsilon^{3/8}} \frac{\phi_{in}^{1/2}(t'\,\varepsilon^{3/8})}{\varepsilon^{3/8}}\,dt' \tag{5.21}$$

The parameter τ_o is chosen such that $\phi_{in}(\tau) > 0$ for $\tau < -\tau_o$. The solutions u_{1i}, u_{2i} are identified as follows: consider the expressions:

$$\begin{pmatrix} \tilde u_1 \\ \tilde u_2 \end{pmatrix}(\tau;\varepsilon) = \frac{3^{1/4}}{\phi_{in}^{1/4}}\begin{pmatrix} \cos \\ -\sin \end{pmatrix}\int_\tau^{-\tau_o} \phi_{in}^{1/2}(\tau';\varepsilon)\,d\tau' \tag{5.22}$$

Then u_{1i}, $u_{2i}(\tau;\varepsilon)$ are the unique solutions of the integral equations:

$$\begin{pmatrix} u_{1i} \\ u_{2i} \end{pmatrix}(\tau;\varepsilon) = \begin{pmatrix} \tilde u_1 \\ \tilde u_2 \end{pmatrix}(\tau;\varepsilon) + \frac{1}{\phi_{in}^{1/4}(\tau)}\int_{-aC_\varepsilon\varepsilon^{-3/8}}^{\tau} R(\phi_{in})\,\phi_{in}^{-1/4}(\tau')\,\sin\left[\int_{\tau'}^{\tau}\phi_{in}^{1/2}(\tau'')\,d\tau''\right] \times \begin{pmatrix} u_{1i} \\ u_{2i} \end{pmatrix}(\tau'';\varepsilon)\,d\tau'' \tag{5.23}$$

with $R(\phi)$ of (3.21).

This Lemma is of importance in the whole domain of parameters $\Delta \prec \Gamma^{1/4}$. It states that, corresponding to the partition of $x_a(t)$ in outer and inner parts, we can represent the solutions of (5.3) as a superposition of an outer (WKB) part and an inner one, in the inner variable τ, eq.(1.11). The inner solutions have a limit as $\varepsilon \to 0$, the outer ones do not. The proof of Lemma 5.3 is straightforward, but not short. We present below its main steps.

Proof: We prove first the statements concerning $u_{1i}, u_{2i}(\tau;\varepsilon)$. To this end, we show first that:

$$\lim_{\varepsilon \to 0} (1 + \tau^{2/3})^{-1} \Phi_{in}(\tau;\varepsilon) = (1 + \tau^{2/3})^{-1} \Phi_{in}(\tau;0) \qquad (5.24)$$

is uniform with respect to τ, as long as $-C_\varepsilon \varepsilon^{-3/8} < \tau < 0$. Indeed, using (4.28), for large τ :

$$\eta_i^Q(\tau) \sim \eta_{oo}(\tau) + \gamma\eta_{o1}(\tau) + \sum_{q=1}^{Q} a_{ooq}\, \tau^{2q+1/3}\, \varepsilon^{3q/4} \sim \eta_{oo}(\tau) + O\left(\gamma\tau^{-4/3}, \tau^{7/3}\varepsilon^{3/4}\right) \qquad (5.25)$$

Thus, for $\tau \in I_\tau(\varepsilon) \equiv \left[-C_\varepsilon \varepsilon^{-3/8}, -\tau_o\right]$, we have the uniform estimate:

$$\eta_i^Q(\tau) = \eta_{oo}(\tau)\,(1 + O(\gamma\tau^{-5/3}, C_\varepsilon^2)) \qquad (5.26)$$

which justifies (5.24). Now, (5.24) implies the estimate:

$$\Phi_{in}^{-1/2}(\tau;\varepsilon)\, R(\Phi_{in}) < \text{const}/\tau^3 \qquad (5.27)$$

independently of ε, for $\tau \in I_\tau(\varepsilon)$. Eqn. (5.23) leads then, with Gronwall's inequality, to:

$$|u_{1i}(\tau) - \tilde{u}_1(\tau)| < \text{const}/\tau^{13/6} \qquad , \qquad (5.28)$$

$\tau \in I_\tau(\varepsilon)$. Taking derivatives in (5.23) with respect to τ, we obtain, using (5.27-28):

$$\left|\frac{du_{1i}}{d\tau} - \frac{d\tilde{u}_1}{d\tau}\right| < \text{const}/\tau^{11/6} \qquad (5.29)$$

$\tau \in I_\tau(\varepsilon)$. Now, as $\varepsilon \to 0$, eqn.(5.22) has a limit $\tilde{u}_{1i}(\tau;0)$ (or $\tilde{u}_{2i}(\tau;0)$) at any finite τ. Also, we can let formally $\varepsilon \to 0$ in (5.23) and obtain the integral equation: (e.g. for u_{1i})

$$u_{1i}(\tau;0) = \tilde{u}_1(\tau;0) + \Phi_{in}^{-1/4}(\tau;0)\int_{-\infty}^{\tau} R(\Phi_{in})\Phi_{in}^{-1/4}(\tau')\,\sin\left[\int_{\tau'}^{\tau}\Phi_{in}^{1/2}(\tau'';0)d\tau''\right]u_1(\tau'')\,d\tau'' \qquad (5.30)$$

Eqn. (5.30) has a unique solution $u_{1i}(\tau;0)$ for $\tau <$ const, as follows from (5.27) and a contraction argument. Subtracting (5.30) from (5.23) we may establish, by means of Gronwall's inequality and of the estimates (5.28-29) that $u_{1i}(\tau;\varepsilon)$ converges to $u_{1i}(\tau;0)$ uniformly on $[-\varepsilon^{-\delta}, \tau_o]$, for a certain $\delta > 0$ and also that $du_{1i}/d\tau\,(\tau;\varepsilon) \longrightarrow du_{1i}/d\tau\,(\tau;0)$ there (uniformly). The restriction to $[-\varepsilon^{-\delta}, \tau_o] \subset I_\varepsilon(\tau)$ is due to the fact that the phases of $\tilde{u}_{1i}(\tau;\varepsilon)$ and $\tilde{u}_{1i}(\tau;0)$

do not approach each other on the whole $I_\varepsilon(\tau)$, as $\varepsilon \to 0$; the condition for this to happen is, clearly: $\varepsilon^{3/4}\ \tau^{2+4/3} \longrightarrow 0$, which means $|\tau| < C\,\varepsilon^{-9/40+\delta_1}$, for some $9/40 > \delta_1 > 0$. With a further restriction on δ_1, we may even ensure that $d\tilde{u}_1/d\tau\ (\tau;\varepsilon)$ approaches $d\tilde{u}_1/d\tau\ (\tau;0)$ on $[-\varepsilon^{-\delta}, -\tau_o]$, $\delta = 9/40 - \delta_1$. This suffices to justify our statements concerning the behaviour of $\tilde{u}_{1i}$, $\tilde{u}_{2i}(\tau;\varepsilon)$ as $\varepsilon \to 0$ on $I_\tau(\varepsilon)$. We may clearly append to $I_\tau(\varepsilon)$ a finite interval $[-\tau_o, 0]$, without impairing on the uniform convergence of u_{1i}, $du_{1i}/d\tau\ (\tau;\varepsilon)$ as $\varepsilon \to 0$.

We evaluate next the difference between:

$$w_{1a}(t) \equiv \chi_{oL}(t;C_\varepsilon)\ u_{1o}(t;\varepsilon) + \chi_i(t;C_\varepsilon)\ u_{1i}(t;\varepsilon) \tag{5.31}$$

and the exact solution $w_1(t)$ of (5.3), with the same initial conditions at $t = -aC_\varepsilon$. If $t < -aC_\varepsilon$, the difference is given by standard WKB estimates: (cf. eqn. (3.21))

$$|w_1(t) - w_{1a}(t)| < \varepsilon^{1/16+1/2}\ \phi^{-1/4}(-aC_\varepsilon)\int_{-5/2}^{-aC_\varepsilon} R(\phi)\,\phi^{1/2}(t')dt' = O(\varepsilon^{9/16}\ C_\varepsilon^{-3/2}) \tag{5.32}$$

In the domain $-aC_\varepsilon < t < -bC_\varepsilon$, the functions $w_{1a}(t)$, $w_{2a}(t)$, eqn. (5.31), satisfy a differential equation of the second order, different (slightly) from (5.3). It contains also a term in dw/dt and we write it as:

$$\frac{d^2w}{dt^2} + P_1(t)\frac{dw}{dt} + P_2(t)\,w = 0 \tag{5.33}$$

Obtaining the expression of $P_1(t)$, $P_2(t)$ in terms of χ_{oL}, χ_i, u_{ko}, u_{ki}, $k=1,2$, requires some straightforward, but lengthy labor: they are close to zero and to $\phi(t)/\varepsilon$ (cf. eqn. (5.3)) in turn. Their departure from these limiting values has two origins: on one hand, the equation satisfied by $u_{1o}(t)$, $u_{2o}(t)$ is not (5.3), but contains WKB corrections (cf. (3.20)); on the other hand, most of the algebraic contributions are proportional to the quality of the matching of u_{1o} with u_{1i} (and of u_{2o} with u_{2i}), as well as of their derivatives up to order three on $[-aC_\varepsilon, -bC_\varepsilon]$. As an example, one term in $P_1(t)$ is:

$$T(t) = W(w_{1a},w_{2a})^{-1}\Big\{\ 2\dot{\chi}_i\chi_o\left[u_{1o}(\dot{u}_{2i} - \dot{u}_{2o}) - u_{2o}(\dot{u}_{1i} - \dot{u}_{1o})\right] + 2\dot{\chi}_o\chi_i \left[-u_{1i}(\dot{u}_{2i} - \dot{u}_{2o}) + u_{2i}(\dot{u}_{1i} - \dot{u}_{1o})\right]\Big\} \tag{5.34}$$

Now, a typical difference $u_{1i} - u_{1o}$ may be estimated as:

$$|u_{1i} - u_{1o}| \leq |u_{1i} - \tilde{u}_1| + |\tilde{u}_1 - u_{1o}| \tag{5.35}$$

with $\tilde{u}_1(t)$ of (5.22). The difference $|u_{1i} - \tilde{u}_1|$ is given by (5.28), with $\tau = C_\varepsilon\varepsilon^{-3/8}$;

it does not depend on the orders K,L,Q of the outer and inner expansions. In view of the choice (5.21) of $\Psi_L(\varepsilon;\tau_o)$, the second difference is proportional to $|\Phi_{in}(\tau;\varepsilon) - \phi(\tau\varepsilon^{3/8})/\varepsilon^{1/4}|$, with τ in $[-a,-b]\ C_\varepsilon\varepsilon^{-3/8}$. The latter difference is $O(\varepsilon^{cP})$, $c>0$ with P of eqn. (4.32) and may be made as small as one wishes by allowing P to be large enough. With this, we verify e.g. $T(t) = O(\varepsilon^{3/4}/C_\varepsilon^3)$, for large P. As a consequence, using:

$$y(t) = w(t)\ \exp\left[-\int_{-aC_\varepsilon}^{t} P_1(t')dt'\right] = w(t)\ (1 + O(\varepsilon^{3/4}/C_\varepsilon^{\ 2})) \tag{5.36}$$

we obtain an equation for y which may be directly compared to (5.3):

$$\frac{d^2y}{dt^2} + \overline{\Pi}(t)\ y = 0 \tag{5.37}$$

where, for P large, using (3.21)

$$\overline{\Pi}(t) = \Phi(t)/\varepsilon - \chi_{oL}(t)\ R(\phi)/\phi + \chi_{oL}(t)\chi_i(t)\ O(\varepsilon^{1/4}/C_\varepsilon^{8/3}) =$$
$$\equiv \phi(t)/\varepsilon + \overline{\Pi}_s(t) \tag{5.38}$$

The second term in (5.38) is $O(C_\varepsilon^{-2})$ and is larger than the third one, the evaluation of which requires most of the labor.

We now rewrite eqn. (5.3) as an integral equation, using the two linearly independent solutions of (5.37):

$$y_{ia}(t) = w_{ia}(t)\ \exp\left[-\int_{-aC_\varepsilon}^{t} P_1(t')\ dt'\right] \qquad , i=1,2 \tag{5.39}$$

and the boundary conditions at $t = -aC_\varepsilon$:

$$w_1(t) = y_{1a}(t) + \int_{-aC_\varepsilon}^{t} K(t;t')\ w_1(t')\ \overline{\Pi}_s(t')\ dt' \tag{5.40}$$

with $K(t;t')$ the same kernel as in (1.22), written with the $y_{ia}(t)$. From eqn. (5.40), we estimate easily, using Gronwall's Lemma:

$$|\ w_1(t) - y_{1a}(t)| = O(\varepsilon^{9/16}/\ C_\varepsilon^{3/2}) \tag{5.41}$$

With (5.39), (5.36), this ends the proof of Lemma 5.3.

Notice, the statement of the Lemma is true for any $\Delta < C\Gamma^{1/4}$, Γ sufficiently large. The reason why we cannot extend it to $t>0$ over this range of Δ is that $x_a(t)$ may acquire a complicated appearance at $t>0$ (see Sect. VII). However, if $C\Gamma^{1/4} < \Delta < B\Gamma^{1/4}$, for some $C>0$, then $x_a(t)$ is still given by (4.30) for all $|t|<\pi/2$ and we may state:

<u>Lemma 5.4</u>: If $C\Gamma^{1/4} < \Delta < B\Gamma^{1/4}$, eqn. (5.3) admits on $[-\pi/2, \pi/2]$ of two linearly independent solutions, of the form: (k=1,2)

$$w_k(t) = \chi_{oL}(t;\tilde{\varepsilon})\, u_{koL}(t;\varepsilon) + \chi_i(t;\tilde{\varepsilon})\, u_{ki}(t;\varepsilon) + \chi_{oR}(t;\tilde{\varepsilon}) u_{koR}(t;\varepsilon) + O(\varepsilon^m) \tag{5.42}$$

where $u_{koL}(t;\varepsilon) = u_{ko}(t;\varepsilon)$ of eqn. (5.19), u_{ki} are solutions of (5.23) and

$$u_{koR}(t;\varepsilon) = 3^{1/4}\, a_k(\varepsilon)\, \varepsilon^{1/16}\, \Phi^{-1/4} \sin\Big[\int_{bC_\varepsilon}^{t} \frac{\Phi(t')^{1/2}}{\varepsilon^{1/2}}\, dt' + \psi_R(\varepsilon;\tau_o) + \psi_k(\varepsilon)\Big] \tag{5.43}$$

In (5.43), the quantities $a_k(\varepsilon), \psi_k(\varepsilon)$ have limits as $\varepsilon \to 0$; $\psi_R(\varepsilon;\tau_o)$ is defined by clear analogy to (5.21); $m = m(\alpha)$ is given in (5.41).

Proof: Using Lemma 4.7, we may repeat the reasoning of Lemma 5.3 for the interval $[0,\pi/2]$. All estimates stay unchanged. Let $w_{kR}(t;\varepsilon)$, $k=1,2$ be the two solutions analogous to (5.18); we denote the latter by $w_{kL}(t;\varepsilon)$, $k=1,2$. Clearly, we may write:

$$w_{kL}(t;\varepsilon) = \sum_{j=1}^{2} a_{kj}(\varepsilon)\, w_{jR}(t;\varepsilon) \tag{5.44}$$

where the matrix $a_{kj}(\varepsilon)$ may be determined by comparing the values of $w_{kL}(t;\varepsilon)$, $w_{jR}(t;\varepsilon)$ at $t = 0$, e.g.

$$a_{12}(\varepsilon) = W(w_{1L},w_{2R}) / W(w_{1R},w_{2R}) \tag{5.45}$$

Now, by Lemma 3.5, $w_{1L}(\tau;\varepsilon)$, $w_{2L}(\tau;\varepsilon)$ have limits at $\tau=0$, together with their derivatives, as $\varepsilon\to 0$. Thus,

$$W(w_{1L},w_{2R}) = \varepsilon^{-3/8}\, k_{12}(\varepsilon) \tag{5.46}$$

where $k_{12}(\varepsilon)$ has a limit as $\varepsilon \to 0$. But $W(w_{1R}, w_{2R}) = W(w_{1L},w_{2L}) = \varepsilon^{-3/8}$ as follows by direct evaluation at $t = bC_\varepsilon$, $t = -aC_\varepsilon$. Writing:

$$a_{k1}(\varepsilon) = a_k(\varepsilon) \sin \psi_k(\varepsilon) \quad , \quad a_{k2}(\varepsilon) = a_k(\varepsilon) \cos \psi_k(\varepsilon) \tag{5.47}$$

$k=1,2$, we obtain the statement of the Lemma.

Comments: (i) The evaluations above also imply $\det|a_{ij}(\varepsilon)| = 1$. With (5.47), we may write:

$$a_1(\varepsilon)\, a_2(\varepsilon) \sin(\psi_1(\varepsilon) - \psi_2(\varepsilon)) = 1 \tag{5.48}$$

which means $a_1(\varepsilon), a_2(\varepsilon) \neq 0$, $\psi_1(\varepsilon) \neq \psi_2(\varepsilon)$, for all ε.

(ii) For large Δ, comparison of (5.43) with (5.12-13) shows that $a_1(\varepsilon) \sim a_2(\varepsilon) \sim s(\varepsilon)/\sqrt{2}$, $\psi_1(\varepsilon) \sim \pi/4 + 1/s^2$, $\psi_2(\varepsilon) \sim \pi/4 - 1/s^2$.

(iii) The proof of Lemma 1.1 proceeds by analogy: an approximate solution to (1.15) may be written similarly to (5.42), where the u_{koL}, u_{koR} are WKB wave functions and the $u_{ki}(\tau;\varepsilon)$ are solutions with prescribed asymptotic behaviour (cf. eqn. (5.22) as $\tau \to -\infty$ of the equation

$$\frac{d^2x}{d\tau^2} + 3\tau^{2/3} x = 0 \quad . \tag{5.49}$$

In computing Floquet exponents (cf. eqn. (2.38)), the only numerical task left is the determination of the transition matrix $a_{ij}(\varepsilon)$ in the potential $3\tau^{2/3}$.

This closes our discussion of the solutions of the variational equation.

VI. Existence and uniqueness of periodic solutions if $A\Gamma^{2/3} > \Delta > C\Gamma^{1/4}$ and existence of some special solutions if $\Delta < C\Gamma^{1/4}$.

We have now gathered all ingredients necessary for the proof of existence and uniqueness of the periodic solutions of (1.3), for sufficiently high damping, for establishing the validity of their (composite) asymptotic expansions of Sect. IV and for the calculation of the Floquet exponents. As in Sect. III, to prove the existence of a periodic solution with the asymptotic expansion $x_a(t)$, eqn. (4.21), we need an estimate of $\|[\mathcal{D}_x(x_a)]^{-1}\|$ (cf. eqns. (1.22) and the definitions (1.19), (1.20)). Only a coarse bound on the latter is needed, since the residuals $\|\mathcal{E}(x_a)\|$ may be made smaller than any power of ε, by allowing sufficiently high order in the expansions.

With this, we may state:

Lemma 6.1: If $A\Gamma^{2/3} > \Delta > \sqrt{3}\,\Gamma^{1/3}$

$$\|[\mathcal{E}_x(x_a)]^{-1}\| = O(\mu^{-1}) \tag{6.1}$$

Proof: This is done by simply inserting expressions (5.4), (5.5) into (1.22) and using:

$$W(x_1,x_2) = O\left(\frac{\mu}{\varepsilon}\right) \exp\left[-\frac{\mu}{\varepsilon}(t + \pi/2)\right] \tag{6.2}$$

as well as the fact that $e_1(t) > k > 0$, (cf. eqn. (5.6)) for all $|t| < \pi/2$. For more accurate estimates, see Ref.[22].

Lemma 6.2: If $\sqrt{3}\,\Gamma^{1/3} > \Delta > B\,\Gamma^{1/4}$, B sufficiently large:

$$\| [\mathcal{D}_x(x_a)]^{-1} \| = 0 \left(\frac{1}{\mu \rho^{1/6}} \right) \tag{6.3}$$

Proof: We use the variable θ, eqn. (5.9), and the WKB wave functions, eqns. (5.12-13). At the turning points, they are bounded from above by $\rho^{-1/6}$ or $\rho^{-1/6}$ s, cf. eqns. (5.11,15). Denoting by $T_i(\theta)$ the analogon of the third term in (1.22), we have the bound:

$$| T_i(\theta)| < \text{const } \frac{\mu^6}{\varepsilon^4} \rho \int_{-\pi\lambda/2}^{\theta} \exp\left[- \frac{\theta - \theta'}{\rho}\right]\left[u_1(\theta') u_2(\theta) - u_1(\theta) u_2(\theta')\right] d\theta' \|f\| \tag{6.4}$$

In deriving (6.4), we have used (5.10) and

$$W_\theta(u_1, u_2) = 1/\rho \tag{6.5}$$

With (5.12), (5.13) , the integrand in (6.4) is O(1), so that $T_i(\theta) = 0(\mu^{-1} \rho^{-1/6})$. Similar bounds hold for the first and second derivatives of $T_i(\theta)$. With (1.23) and recalling the factors ε, μ present in the norm (1,19), we justify (6.3) completely.

Using Lemma 5.4, we can establish similarly:

Lemma 6.3: If $B \Gamma^{1/4} > \Delta > C \Gamma^{1/4}$, for any $C > 0$, then, for large Γ :

$$\| [\mathcal{D}_x(x_a)]^{-1} \| = 0(\varepsilon^{-5/8}) \tag{6.6}$$

The proof is immediate using the wave functions (5.18).

With this, and using the residuals $\mathcal{D}(x_a)$ in eqns. (4.23), (4.32), Newton's method described in Section III establishes the existence, in each of the parameter ranges described above, for large Γ, of an odd periodic solution $x_p(t)$ of (1.3), which is approximated better than any given power of ε by $x_a(t)$, eqns. (4.21), (4.34), provided only the order P of the expansion is large enough. With the choice of the norm (1.19), the same is true concerning the approximation of the first and second derivatives of $x_p(t)$ by dx_a/dt, d^2x_a/dt^2. Because $x_a(t) \sim (\sin t)^{1/3}$ for t away from $n\pi$, in our range of parameters (cf. (4.1),(4.2)), it follows that $x_p(t)$ possesses property (H1) of Section II.

Now, the solutions of the variational equation (1.14) associated to $x_p(t)$ and their derivatives differ from those of eqn. (5.1) by quantities of $O(\varepsilon^s)$, where s may be made large, by choosing P large. This follows by rewriting (1.14) with the help of two solutions $x_1(t)$, $x_2(t)$ of (5.1) as:

$$x_{1p}(t) = x_1(t) + \int_{-\pi/2}^{t} \frac{x_1(t')\, x_2(t) - x_1(t)\, x_2(t')}{\varepsilon\, W(t')}\; 3\, x_{1p}(t')\left[x_p^2(t') - x_a^2(t')\right] dt' \tag{6.7}$$

We use then the bounds derived above on the kernel of (6.7), the fact that $\sup_t |x_p - x_a|$ is $O(\varepsilon^s)$ and Gronwall's Lemma to obtain the announced statement. Thus, the Floquet multipliers may be computed accurately using the wave functions of Sect. V, solutions of (5.1).

If $A\Gamma^{2/3} > \Delta > D_1\Gamma^{1/3}$, D_1 sufficiently large, eqns. (5.5-5.6) yield, via formulae (2.23):

$$\tilde{\mu}_1 \equiv e^{-\lambda_1 \pi} \simeq \text{const}\ \exp\left[-\frac{3}{2\mu}\int_{-\pi/2}^{\pi/2} x_p^2(t')\,dt'\right] \simeq \text{const}\ \exp\left[-\frac{3}{2\mu}\int_{-\pi/2}^{\pi/2}(\sin t)^{2/3}\,dt\right] \tag{6.8}$$

$$\tilde{\mu}_2 \equiv e^{-\lambda_2 \pi} \simeq \text{const}\ \exp\left[-\frac{\mu}{\varepsilon}\int_{-\pi/2}^{\pi/2}(1 + (1 - 3\gamma x_p^2)^{1/2}\,dt\right] \simeq \text{const}\ \exp\left[-\frac{2\mu}{\varepsilon}\pi\right] \tag{6.9}$$

If $D_2\Gamma^{1/3} > \Delta > B\Gamma^{1/4}$, B large, eqns. (5.2), (5.12),(5.13) yield:

$$\tilde{\mu}_{1,2} \simeq \exp\left[-\frac{\mu}{\varepsilon}\pi\right]\left\{\cosh[\ln s(\rho)]\cos\left(\frac{\Psi_R+\Psi_L}{\rho}\right) \pm \left[\cosh^2 \ln s(\rho)\cos^2\left(\frac{\Psi_R+\Psi_L}{\rho}\right) - 1\right]^{1/2}\right\} \tag{6.10}$$

with $s(\rho)$ of (5.15), $\Psi_L = \Psi(-\pi/2, \theta_L)$, $\Psi_R = \Psi(\theta_R, \pi/2)$ (cf. (5.14)). For large Γ:

$$\frac{\Psi_R + \Psi_L}{\rho} \equiv \frac{\Psi_T}{\sqrt{\varepsilon}} \sim \frac{1}{\sqrt{\varepsilon}}\int_{-\pi/2}^{\pi/2} |\sin t|^{1/3}\,dt \tag{6.11}$$

Eqn. (6.10) shows a sequence of equidistant maxima (and minima) in $\Gamma^{1/3}$, spaced by the same $\delta(\Gamma^{1/3})$ as in (1.16) and of height $\exp(-\mu\pi/\varepsilon)\cdot s$ ($\exp(-\mu\pi/\varepsilon)/s$ in turn), over most of the interval of length $\delta(\Gamma^{1/3})$, the exponents are real; there are narrow spikes around the places where $\Psi_R + \Psi_L = n\pi\rho$; there $|\lambda_1| = |\lambda_2| = \exp(-\mu\pi/\varepsilon)$.

If $B\Gamma^{1/4} > \Delta > C\Gamma^{1/4}$, for some $C > 0$, eqns. (5.42) lead to:

$$\tilde{\mu}_{1,2}(\varepsilon) = \exp\left(\frac{\mu}{\varepsilon}\pi\right) / \left\{\frac{b(\varepsilon)}{2}\sin\left(\frac{\Psi_T}{\sqrt{\varepsilon}} + c(\varepsilon)\right) \pm \left[\frac{b^2(\varepsilon)}{4}\sin^2\left(\frac{\Psi_T}{\sqrt{\varepsilon}} + c(\varepsilon)\right) - 1\right]^{1/2}\right\} \tag{6.12}$$

where (cf.(5.43)):

$$b^2(\varepsilon) = a_1^2(\varepsilon) + a_2^2(\varepsilon) - 2 \tag{6.13}$$

and $c(\varepsilon)$ depends on $\Psi_1(\varepsilon), \Psi_2(\varepsilon)$, eqn. (5.43); by Lemma 5.4, these quantities have limits as $\varepsilon \to 0$. For large $\Delta/\Gamma^{1/4}$, $b(\varepsilon) \sim s(\rho)\sqrt{2}$, $c(\varepsilon) \simeq O(1/s^2) + \pi/2$. Eqn. (5.48) implies that $b^2(\varepsilon) \geq 0$; if $|b(\varepsilon)| > 2$, there are again equidistant maxima of $\max_{1,2}|\tilde{\mu}_{1,2}(\Gamma^{1/3})|$, with the same spacing (1.16), separated by intervals where $|\tilde{\mu}_1(\varepsilon)| = |\tilde{\mu}_2(\varepsilon)| = \exp(-\mu\pi/\varepsilon) \equiv \exp(-\pi\Delta)$.

We establish now the uniqueness of the periodic solutions constructed above, following the pattern outlined in Section II. We choose $T = t_o + \pi$ (n=1) in the formulae there.

If $A\Gamma^{2/3} > \Delta > D_1 \Gamma^{1/3}$ ($D_1 > \sqrt{3}$), we choose in (2.25) $\lambda(\varepsilon) = (k/A)\int_{-\pi/2}^{\pi/2} x_p^2(t')dt'$ with $k < 1$ independent of ε. Clearly, since $x_p(t) \sim (\sin t)^{1/3}$, $\lambda(\varepsilon) < \lambda_o < \lambda_1(\varepsilon)$, eqn. (6.8), for small ε, with λ_o independent of ε. With (5.4),(5.5) and the choice of λ above, we may let in eqn. (2.32) $\|v_i\|_\lambda = |v_i(t_o)| = 1$, $\|\dot{v}_i\|_\lambda = |\dot{v}_i(t_o)| = (1/\mu, \mu/\varepsilon)$, i=1,2. As in (2.36), we let $g(\varepsilon) = \varepsilon^{1/2}$ so that $T(\varepsilon) = (1 + \sqrt{\varepsilon}/\mu)^2$. From (2.33), we conclude that if:

$$|u(t_o)| + \varepsilon^{1/2}|\dot{u}(t_o)| < M < \frac{\mu}{3} \frac{1}{(1 + \sqrt{\varepsilon}/\mu)^2} (C_1 + \varepsilon^{1/2} C_2)^{-1} \tag{6.14}$$

and if

$$(1 + \sqrt{\varepsilon}/\mu)^2 \exp(-\lambda_o \pi) < 1 \tag{6.15}$$

all solutions of (2.12) starting in $\mathcal{U}(M)$ approach zero. Since $\sqrt{\varepsilon}/\mu = \Gamma^{1/3}/\Delta < 1/D_1$,(6.15) can be fulfilled in our range of parameters. Now, Lemma (2.3) shows that, in an interval $[t_1 + \delta, t_2]$, $0 < t_1 < t_2 < \pi$ (mod π):

$$|u(t)| + (\sqrt{\varepsilon}/\mu)\mu|\dot{u}(t)| < \text{const} \exp[-C(t - t_1)/\mu] \tag{6.16}$$

Thus, if we let A be small enough, we find for any solution of (2.12) a time t_o in $[t_1, t_2]$ so that (6.14) is fulfilled. This proves uniqueness for the present region of parameter space.

If $D_2 \Gamma^{1/3} > \Delta > B\Gamma^{1/4}$, $D_2 < 3$, B large, we choose again $\lambda = \lambda_o$, independent of ε and such that

$$\lambda_o < \frac{\mu}{\varepsilon} - \frac{1}{\pi} \ln s(\rho) \leq -\frac{1}{\pi} \ln[\min(\tilde{\mu}_1, \tilde{\mu}_2)(\varepsilon)] \quad , \tag{6.17}$$

cf.eqns. (5.15),(6.10). We notice that, as $\varepsilon \to 0$, the upper bound on λ_o recedes to infinity. Further, since in our range of parameters:

$$\frac{1}{\rho}(1 - Q)^{1/2} < \mu/\varepsilon \qquad , t \in [t_L, t_R] \tag{6.18}$$

the wave functions $v_1(t)$, $v_2(t)$ are monotonically decreasing, apart from neighbourhoods of the turning points where they are bounded from above by $\rho^{-1/6}$ (cf. eqn.(5.16)). With this, one can verify that, in eqn. (2.23), $\|K\| < \rho^{-1/3} = K_B(\varepsilon;\lambda)$. For Γ large enough, it is true that $\|v_i\|_\lambda = v_i(t_o) \sim (\mu^2/\varepsilon)^{1/4}$ and $\|\dot{v}_i\|_\lambda \sim \dot{v}_i(t_o) \sim (\varepsilon/\mu^2)^{7/4}/\rho$. It follows that $K_D(\varepsilon;\lambda) = \rho^{-7/6}(\varepsilon/\mu^2)^{3/2}$ and, with the choice of (2.36), $g(\varepsilon) = \rho(\mu^2/\varepsilon)^2 = \varepsilon^{1/2}$, $T(\varepsilon) = 4$. From (2.33), it follows that, if

$$|u(t_o)| + \sqrt{\varepsilon}\,|\dot{u}(t_o)| < M < \text{const}\ \mu\rho^{4/3} \tag{6.19}$$

and if

$$4\,e^{-\lambda_o \pi} < 1 \tag{6.20}$$

all solutions of (2.12) starting in $\mathcal{U}(M)$ vanish as $t \to \infty$. Clearly, (6.20) may be satisfied letting λ_o be large enough (but finite), which is possible for ε small enough. On the other hand, Lemma 2.2 ensures that, for $t \in [t_1, t_2]$, $0 < t_1 < t_2 < \pi \pmod{\pi}$:

$$|u(t)| + \sqrt{\varepsilon}\,|\dot{u}(t)| < K \exp[-C\tfrac{\mu}{\sqrt{\varepsilon}}(t - t_1)] \tag{6.21}$$

which shows the existence of a $t_o \neq n\pi$, where (6.19) holds, for ε small enough. This proves the uniqueness of $x_p(t)$ in this range of parameters.

If $B\Gamma^{1/4} > \Delta > C\Gamma^{1/4}$, for some $C > 0$, we choose $\lambda = \lambda_o < \frac{\mu}{\sqrt{\varepsilon}} - \frac{1}{\pi}\ln(b(\varepsilon))$ (cf. eqn. (6.13-14)) and $g(\varepsilon) = \sqrt{\varepsilon}$. Using eqns. (5.42), we obtain easily $K_B(\varepsilon;\lambda) = C_1$, $K_D(\varepsilon;\lambda) = C_2\varepsilon^{-3/8}$, so that the version of (2.33) of interest reads:

$$|u(t_o)| + \sqrt{\varepsilon}\,|\dot{u}(t_o)| < M < \text{const}\ \varepsilon^{5/8} \tag{6.22}$$

as is expected from (6.19). The arguments used in the previous parameter range stay now unchanged. With this, we may conclude:

<u>Theorem 6.1</u>: If $A\Gamma^{2/3} > \Delta > C\Gamma^{1/4}$, for any $C > 0$, Duffing's equation (1.3) admits of a unique periodic solution $x_p(t;\Gamma)$, if Γ is large enough. This solution has an asymptotic expansion to order ε^N, for any $N > 0$, uniform on $[-\pi/2, \pi/2]$, given by (4.21) (or (4.30)), where the integers Q,K,L are chosen appropriately large.

In the remaining part of the paper, we discuss only the situation $\Delta < C\Gamma^{1/4}$. The solution of the variational equation in Lemma 5.3 and the estimates of residuals in Lemmas 4.8, 4.9 allow us to obtain two remarkable solutions $x_L(t)$, $x_R(t)$ of Duffing's equation (1.3), valid for any $\Delta < C\Gamma^{1/4}$ and Γ large, however only for $-\pi/2 < t < 0$ (for $x_L(t)$) and for $0 < t < \pi/2$ (for $x_R(t)$). We state first:

<u>Lemma 6.4</u>: Eqn. (1.3) admits of a solution $x_L(t;\varepsilon)$, which is uniformly approximated, together with its first two derivatives by $x_{La}(t)$, eqn. (4.31), arbitrarily well as $\varepsilon \to 0$, if K,L,Q are large enough, and which obeys the

initial conditions: $x_L(-\pi/2) = x_{La}(-\pi/2)$, $dx_L/dt(-\pi/2) = dx_{La}/dt(-\pi/2)$.

Clearly, the solution so singled out depends on the approximant that was chosen. The proof is done again by Newton's method, this time in a space $\mathbb{D}(-\pi/2,0)$ of twice differentiable functions on $[-\pi/2, 0]$, with the same norm (1.19), but vanishing and with vanishing derivative at $t = -\pi/2$. Writing:

$$x(t) = x_a(t) + r(t) \tag{6.23}$$

with $r(-\pi/2) = \dot{r}(-\pi/2) = 0$, the Duffing eqn. (1.3) maps $\mathbb{D}(-\pi/2,0)$ onto $\mathbb{C}(-\pi/2,0)$, without any restriction at $t = -\pi/2$. The inverse Fréchet derivative at x_{aL} exists and is given by (1.22) with $a_1 = a_2 = 0$. Using eqns. (5.18), we verify easily, for $\Delta < C\Gamma^{1/4}$:

$$\| [\mathcal{D}_x(x_{aL})]^{-1} \| = O(\varepsilon^{3/8}/\mu) = O(\varepsilon^{-1/4}) \tag{6.24}$$

The estimate (4.32) and the standard reasoning of Sect. III, eqns. (3.7-11), establish then Lemma 6.4.

The same reasoning, using the estimate (4.42) and the (obvious) analogon of (5.18) for the interval $[0, \pi/2]$, shows:

<u>Lemma 6.5</u> : Duffing's equation (1.3) admits of a solution $x_R(t;\varepsilon)$, which is uniformly approximated, together with its first two derivatives, by $x_{aR}(t)$, eqn. (4.41), on $0 \leq t \leq \pi/2$, arbitrarily well (to $O(\varepsilon^{cP(S)})$) if the integers P(S), eqn. (4.42) are sufficiently large, and which obeys $x_R(0;\varepsilon) = x_{aR}(0;\varepsilon)$, $\dot{x}_R(0;\varepsilon) = \dot{x}_{aR}(0;\varepsilon)$.

With the help of the solutions $x_{L,R}(t;\varepsilon)$, we may construct a uniform approximant to an odd periodic solution of (1.3), even if $\Delta < C\Gamma^{1/4}$. We turn now to this.

VII. A special solution of Duffing's equation for $t > 0$; $\Delta < C\Gamma^{1/4}$.

The two solutions $x_L(t;\varepsilon)$, $x_R(t;\varepsilon)$ of Section VI assume at $t = 0$ different values. If we continue $x_L(t;\varepsilon)$ past $t = 0$, we expect first an oscillatory behaviour, which dies out in a time $t \simeq \varepsilon/\mu \doteq 1/\Delta$. If $1/\Delta \to 0$ as $\Gamma \to \infty$, the solution settles down near $x_R(t;\varepsilon) \sim (\sin t)^{1/3}$, for $t > t_o$, for any finite $t_o > 0$.

Clearly, for small ε, μ, $x_R(\pi/2) \simeq - x_L(-\pi/2)$, $\dot{x}_R(\pi/2)$ $- \dot{x}_L(-\pi/2)$. Thus the continuation of $x_L(t;\varepsilon)$ to the whole interval $[-\pi/2, \pi/2]$ may offer a uniform approximation $x_a(t)$ over this interval to an odd periodic solution of (1.3). We make now these considerations precise.

We write:

$$\begin{aligned} x_a(t) &= x_L(t) && \text{if } t<0 \\ x_a(t) &= x_R(t) + v(t) && \text{if } t>0 \end{aligned} \tag{7.1}$$

where v(t) is the solution of:

$$\varepsilon \ddot{v} + 2\mu \dot{v} + 3 x_R^2 v + 3 x_R v^2 + v^3 = 0 \tag{7.2}$$

which obeys at t=0:

$$\begin{aligned} v(0) &= x_L(0) - x_R(0) \\ \dot{v}(0) &= \dot{x}_L(0) - \dot{x}_R(0) \end{aligned} \tag{7.3}$$

The solution $v(t;\varepsilon)$ depends on ε not only through the parameters of (7.2), but also through $x_R(t;\varepsilon)$, $x_L(t;\varepsilon)$. We are interested in the behaviour of v(t) for $t > C\,\varepsilon^{3/8}$ (Recall $\varepsilon^{3/8} \prec \varepsilon/\mu$ in our range of parameters).

To this end, we change variables to:

$$\tau = t\,\varepsilon^{-3/8} \qquad , \qquad u(t) = v(t)\,\varepsilon^{-1/8} \tag{7.4}$$

and denote:

$$\eta(\tau) \equiv x_R(\tau)\,\varepsilon^{-1/8} \qquad\qquad \gamma = \mu/\varepsilon^{5/8} = \rho^{-1/4} \tag{7.5}$$

so that (7.2) becomes:

$$\frac{d^2u}{d\tau^2} + 2\gamma \frac{du}{d\tau} + 3\eta^2(\tau)\,u + 3\eta(\tau)\,u^2 + u^3 = 0 \tag{7.6}$$

Clearly, eqn. (7.6) has a τ scale $1/\gamma$ for the decay of oscillations. We shall derive in the following approximants to the solutions of (7.6) for $\tau \in I_\tau(\varepsilon) \equiv [\tau_o, C_\varepsilon\,\varepsilon^{-3/8}] \equiv - I_\tau(\varepsilon)$ of Sect. V, eqns. (5.25) ff.), with τ_o sufficiently large and $C_\varepsilon \to 0$ as $\varepsilon \to 0$ such that:

$$C_\varepsilon\,\varepsilon^{-3/8} \succ 1/\gamma \tag{7.7}$$

Eqn. (7.6) is further transformed through the change of variables:

$$\xi = \int_{\tau_o}^{\tau} \sqrt{3}\,\eta(\tau')\,d\tau' \qquad , \qquad u = p(\xi)\,\eta^{-1/2} \tag{7.8}$$

and becomes:

$$\frac{d^2p}{d\xi^2} + 2\gamma\,\frac{h_o(\xi)}{\xi^{1/4}}\,\frac{dp}{d\xi} + p\left[1 + K(\xi;\gamma)\right] + \frac{p^2}{\xi^{3/8}}\,h_1(\xi) + \frac{p^3}{3\,\xi^{3/4}}\,h_2(\xi) = 0 \tag{7.9}$$

where (i=0,1,2):

$$h_i(\xi) = C_i(1 + O(\frac{1}{\xi^2}, \xi^{3/4}\varepsilon^{3/2}, \frac{\gamma}{\xi^{5/4}})) \equiv C_i(1 + O(r(\xi)) \qquad (7.10)$$

and

$$C_o = \frac{1}{\sqrt{3}}\omega_o^{1/4}, \quad C_1 = \omega_o^{3/8}, \quad C_2 = \omega_o^{3/4} \qquad (7.11)$$

with $\omega_o = 3\sqrt{3}/4$. Further:

$$K(\xi;\gamma) = \left[\frac{7\omega_o^2}{108}\frac{1}{\xi^2} - \frac{\gamma}{9}\frac{\omega_o^{5/4}}{\xi^{5/4}}\right](1 + O(r(\xi))) \qquad (7.12)$$

The notation $O(r(\xi))$ means " $<$ const max $(\xi^{-2}, \varepsilon^{3/4}\xi^{3/2}, \gamma\xi^{-5/4})$ ", for $\xi \in I_\xi(\varepsilon)$:

$$I_\xi(\varepsilon) \equiv [\xi_o, C_\varepsilon^{3/4}\varepsilon^{-1/2}] \qquad (7.13)$$

In deriving (7.9-12), we made use of Lemma 6.5 and of the estimate (5.25).

The behaviour of the solutions of (7.9) for large ξ, ε fixed, is studied by means of a sequence of transformations, similar to those of the method of averaging (Refs.[25,26]). As in the latter, the justification of the results depends on an a priori bound on the solutions of (7.9). This is afforded by:

Lemma 7.1: If ε is sufficiently small, the solutions of (7.9) are bounded on $[0, C_\varepsilon^{3/4}\varepsilon^{-1/2}]$, together with their derivatives, uniformly with respect to ε.

The phrase "uniformly with respect to ε" means that all solutions whose initial conditions at $\xi = \xi_o$ lie in a certain bounded set D of $\mathbb{R}^2$, independent of ε, are also bounded on $I_\xi(\varepsilon)$ (with their derivatives), independently of ε.

The proof of Lemma 7.1 is displayed in Appendix A.

To proceed, we introduce polar coordinates in eqn. (7.9):

$$p = R\cos(\xi + \phi)$$
$$\frac{dp}{d\xi} = -R\sin(\xi + \phi) \qquad (7.14)$$

and obtain:

$$\frac{dR}{d\xi} = -\frac{2\gamma}{\xi^{1/4}}h_o(\xi)R\sin^2(\xi+\phi) + \frac{R}{2}\sin 2(\xi+\phi)\left[K(\xi) + \frac{R\,h_1(\xi)\cos(\xi+\phi)}{\xi^{3/8}} + \frac{R^2\,h_2(\xi)\cos^2(\xi+\phi)}{3\xi^{3/4}}\right] \qquad (7.15a)$$

$$\frac{d\phi}{d\xi} = -\frac{\gamma}{\xi^{1/4}}h_o(\xi)\sin 2(\xi+\phi) + \cos^2(\xi+\phi)\left[K(\xi) + \frac{R\,h_1(\xi)\cos(\xi+\phi)}{\xi^{3/8}} + \frac{R^2\,h_2(\xi)\cos^2(\xi+\phi)}{3\xi^{3/4}}\right] \qquad (7.15b)$$

Let $R^{(o)}$ be the bound offered by Lemma 7.1 on $R(\xi)$ for $\xi > 0$. We perform now a transformation of the "averaging" type, to remove terms in $\xi^{-3/8}$ in (7.15):

$$R_1 = R + \frac{R^2}{3} \frac{\cos^3(\xi+\phi)\, h_1(\xi)}{\xi^{3/8}} \tag{7.16a}$$

$$\phi_1 = \phi - \frac{R}{\xi^{3/8}} h_1(\xi) \sin(\xi+\phi)\left[1 - \frac{\sin^2(\xi+\phi)}{3}\right] \tag{7.16b}$$

At a fixed value of ξ, the transformation (7.16) is one - to - one from the strip $0 < R < R^{(o)}$ in the (R,ϕ) plane onto its domain of values, provided ξ is large enough (depending on $R^{(o)}$). This one sees as follows: (cf. also Ref.[27],p.294) if two points (R,ϕ), (R',ϕ') are mapped by (7.16) onto the same point (R_1, ϕ_1), eqn. (7.16b) implies that:

$$|\phi - \phi'| < \text{const}|R - R'|\, \xi^{-3/8} \tag{7.17}$$

But (7.17) and the requirement $R_1(R,\phi) = R_1(R',\phi')$ are contradictory if $R,R' < R^{(o)}$ and ξ is large enough, say $\xi > \xi_o$. Further, eqn. (7.16a) shows that the image of the strip $0 < R < R^{(o)}$ is contained in $0 < R_1 < R^{(o)} + R^{(o)2}\, h_1(\xi)/(3\xi^{3/8})$ and one may verify that $\xi_o(R^{(o)})$ may be chosen such that, for $\xi > \xi_o$, this image covers $0 < R_1 < R^{(o)} - k_1 R^{(o)}/\xi^{3/8}$ for some $k_1 > 0$ (one solves (7.16a-b) by Newton's method).

The transformation (7.16) changes (7.15) into ($\psi = \xi + \phi$):

$$\frac{dR_1}{d\xi} = -\frac{2\gamma}{\xi^{1/4}} R\, h_o(\xi) \sin^2\psi + \frac{R^3 h_2(\xi)}{3\xi^{3/4}} \cos^3\psi \sin^3\psi + O\!\left(\frac{R^4}{\xi^{9/8}}, RK(\xi), \frac{R^2\gamma}{\xi^{5/4}}, R^2 \varepsilon^{3/4} \xi^{-5/8}\right) \tag{7.18a}$$

$$\frac{d\phi_1}{d\xi} = -\frac{\gamma}{\xi^{1/4}} h_o(\xi) \sin 2\psi - \frac{2R^2}{3\xi^{3/4}} h_2(\xi) \cos^2\psi\, (\sin^2\psi + \cos^4\psi) + O\!\left(\frac{R^3}{\xi^{9/8}}, K(\xi), \frac{\gamma R}{\xi^{5/4}}, R\,\varepsilon^{3/4} \xi^{-5/8}\right) \tag{7.18b}$$

Eqn. (7.18b) contains resonant terms falling off like $\xi^{-3/4}$. All terms under the order sign in (7.18) have trigonometric factors with zero average over a interval of length 2π, if ϕ is held fixed. The averaged equation for $\phi_1(\xi)$ reads:

$$\frac{d\bar{\phi}_1}{d\xi} = -\frac{7}{24} \frac{R^2}{\xi^{3/4}} h_2(\xi) \tag{7.19}$$

We may perform another transformation $(R_1, \phi_1) \longrightarrow (R_2, \phi_2)$, similar to (7.15), to remove from (7.18) the nonresonant terms with a falloff like $\xi^{-3/4}$. It turns out that the resulting equations have no resonant terms in $\xi^{-9/8}$; thus, terms in

$\xi^{-9/8}$ can be removed through a further transformation. The leading terms not involving γ are then $O(\xi^{-11/8})$. We proceed then to removing the nonresonant terms containing γ . A general transformation $(R_1, \phi_1) \quad (R_{1+1}, \phi_{1+1})$ may be written as:

$$R_{1+1} = R_1 + f_1(R, \xi+\phi; \xi) \tag{7.20a}$$

$$\phi_{1+1} = \phi_1 + g_1(R, \xi+\phi; \xi) \tag{7.20b}$$

The computation of $dR_{1+1}/d\xi$, $d\phi_{1+1}/d\xi$ involves only $dR_1/d\xi$, $d\phi_1/d\xi$, which are in turn expressed in terms of the original variables R, ϕ and the quantities $dR/d\xi$, $d\phi/d\xi$, available from (7.15). Thus, new resonant terms may appear only through the combination of the trigonometrical factors of the transformation (of f_1, g_1) with those of eqn. (7.15). As will be apparent below, precise algebraic results are needed only for the first few steps. Let the index L stand for "last" ; we obtain:

$$\frac{dR_L}{d\xi} = -\frac{\gamma}{\xi^{1/4}} R\, h_o(\xi) + O\left(\frac{\gamma R^3}{\xi}, \frac{\gamma^3 R}{\xi^{3/4}}, R\, g_1(\xi;R)\right) \tag{7.21}$$

$$\frac{d\phi_L}{d\xi} = -\frac{7}{24}\frac{R^2}{\xi^{3/4}} h_2(\xi) + O\left(\frac{\gamma^2}{\xi^{1/2}}, \frac{\gamma R^2}{\xi}, g_2(\xi;R)\right) \tag{7.22}$$

where the $g_i(\xi;R)$, $i=1,2$, are polynomials of low order in R with coefficients of $O(\xi^{-11/8}, \gamma\xi^{-9/8}, \varepsilon^{3/4}\xi^{-5/8})$.

From $R(\xi) < R^{(o)}$, it follows that $R_L(\xi) < R^{(o)}(1 + k\xi_o^{-3/8}) \equiv R_L^{(o)}$, for some $k > 0$. Further, we may find $\xi_o(R^{(o)})$ (possibly larger than before), so that, for $\xi > \xi_o$, the compound transformation $(R,\phi) \rightarrow (R_L, \phi_L)$ may be inverted on $R_L < R_L^{(o)}$. It is true that:

$$R = R_L + O(R_L^2\,\xi^{-3/8}, \gamma R_L \xi^{-1/4}) \tag{7.23}$$

and similarly for ϕ .

We can use now eqn. (7.21), with $R = R(R_L, \phi_L)$, to place a much tighter bound on the behaviour of $R_L(\xi)$ on $I_\xi(\varepsilon)$, eqn. (7.13). To this end, using (7.23), we separate in (7.21) terms that are linear in R_L and let:

$$s(\xi) = \int^{\xi}\left[\frac{\gamma h_o(\xi')}{(\xi')^{1/4}} + \text{const}\frac{\gamma^3}{(\xi')^{3/4}} + g_{1o}(\xi')\right](1 + O(\gamma(\xi')^{-1/4}))\, d\xi' =$$
$$= \gamma\tau(\xi) + O(\gamma^2\tau^{2/3})$$

We multiply (7.21) by $\exp[s(\xi)]$, let

$$F(\xi) = R_L(\xi)\exp[s(\xi)] \tag{7.25}$$

use, for $\xi > \xi_o(R^{(o)})$, the inequality (cf.(7.23)):

$$R < R_L\left[1 + k_1(\xi^{-3/8} + \gamma\xi^{-1/4})\right] \tag{7.26}$$

with $k_1 = k_1(R^{(o)})$ and transform (7.21) into the integral inequality:

$$F(\xi) \leq F(\xi_o) + C_1\int_{\xi_o}^{\xi}\gamma\frac{F^3}{\xi'}\,e^{-2s(\xi')}d\xi' + \int_{\xi_o}^{\xi}F^2\,e^{-s(\xi')}\,\tilde{g}_1(\xi';\,Fe^{-s(\xi')})\,d\xi'$$
$$= I(\xi;\xi_o;F) \tag{7.27}$$

where $\tilde{g}_1$ is a polynomial in Fe^{-s}, of one degree less than g_1 of (7.21) and with positive coefficients of the same order of magnitude. Now, by Lemma 7.1:

$$|F(\xi_o)| < R_L^{(o)}\exp s(\xi_o) \tag{7.28}$$

We observe next that the Volterra integral equation obtained by placing an equality sign in (7.27) admits, for ξ_o large enough and ε small enough, of a bounded solution on $I_\xi(\varepsilon)$. Indeed, $I(\xi;\xi_o;F)$ of (7.27) maps a ball of radius $(1+\delta)C\exp[s(\xi_o)]$ in the space of functions continuous on $I_\xi(\varepsilon)$ into itself, provided ξ_o, ε are chosen so that quantities like: $\xi_o^{-3/8}\exp[s(\xi_o)]$, $\gamma(\ln\xi_o)\exp[-\gamma s(\xi_o)]$, $\gamma\exp[s(\xi_o)]\ln(1/\gamma)$, $\varepsilon^{9/16}C_\varepsilon^{9/32}$ are sufficiently smaller than unity. The contraction condition is obviously satisfied in the same manner.

We call $F_o(\xi;\varepsilon)$ the (unique) solution of this equation; it is clearly positive on $I_\xi(\varepsilon)$, since all terms in $I(\xi;\xi_o;F)$ are positive. The monotonicity and Lipschitz continuity of the integrand in (7.27) with respect to F for positive F imply that all positive solutions of (7.27) obey:

$$F(\xi;\varepsilon) \leq F_o(\xi;\varepsilon) = F(\xi_o) + I(\xi;\xi_o;F_o) \tag{7.29}$$

for all $\xi \in I_\xi(\varepsilon)$ (see Ref.[28],p.18). We denote $\xi_R \sim C_\varepsilon^{3/4}\,\varepsilon^{-1/2}$, the right end of $I_\xi(\varepsilon)$ and $F(\xi_R) = R_{oL}(\varepsilon)$. Clearly, $R_{oL}(\varepsilon)$ depends on the initial conditions.

Integration of (7.21) from ξ_R backwards and use of (7.29) leads to the inequalities valid for all $\xi \in I_\xi$:

$$R_{oL}(\varepsilon) - I(\xi_R;\xi;F_o) \leq F(\xi) \leq R_{oL}(\varepsilon) + I(\xi_R;\xi;F_o) \tag{7.30}$$

With (7.23), (7.25) and the bound on $|F_o(\xi)|$ above, this means:

$$R(\xi;\varepsilon) = \left[R_{oL}(\varepsilon) + O(\xi^{-3/8}, \gamma\ln\frac{1}{\gamma})\right]\exp[-s(\xi)] \tag{7.31}$$

where the second estimate ($\gamma\ln 1/\gamma$) is valid if $\xi \prec 1/\gamma^{4/3}$.

We turn now to eqn. (7.22), use (7.31) for $R(\xi;\varepsilon)$ and integrate from ξ_R to ξ; we obtain:

$$\Phi_L(\xi) = \Psi_{oL}(\varepsilon) - \frac{7}{24} R_{oL}^2 J(\gamma;\xi;3/4) + \text{const}\,\gamma^2 \xi^{1/2} + O(\xi^{-1/8}, \gamma^{2/3}\ln\frac{1}{\gamma}) \tag{7.32}$$

where we have denoted:

$$\Psi_{oL}(\varepsilon) = \Phi_L(\xi_R) + \frac{7}{24} R_{oL}^2 J(\gamma;\xi_R;3/4) - \text{const}\,\gamma^2 \xi_R^{1/2} \tag{7.33}$$

and

$$J(\gamma;\xi;k) \equiv \int_{\xi_o}^{\xi} \frac{e^{-2s(\xi')}}{(\xi')^k} h_2(\xi')\,d\xi' \sim \int_{\tau_o}^{\tau} \frac{e^{-2\gamma\tau'}}{(\tau')^{(4k-1)/3}}\,d\tau' = O(\tau_o^{4(1-k)/3}),\ \text{if } k>1,$$
$$= O(\ln\frac{1}{\gamma}),\ \text{if } k=1, \quad = O(\gamma^{4(k-1)/3}),\ \text{if } k<1 \,. \tag{7.34}$$

Eqn. (7.34) is also used to obtain the $O(\cdot)$ term in (7.32).

The behaviour of the phase $\Phi_L(\xi(\tau))$ is interesting: on a τ scale short compared to $1/\gamma$, its absolute value increases like $\tau^{1/3}$, due to the second term in (7.32); at $\tau \sim 1/\gamma$, it reaches a magnitude $O(\gamma^{-1/3})$ and levels off. The contribution of the term in (7.32) proportional to γ^2 is only $O(\gamma^{4/3})$ at $\tau \sim 1/\gamma$ and if $\mu \prec \varepsilon^{3/4}$ ($\Delta \prec \Gamma^{1/6}$), it vanishes as $\varepsilon \to 0$ even at $\tau \sim \varepsilon^{-3/8}$. Even otherwise, its effect is negligible due to the smallness of $R(\xi)$ at $\tau \sim \gamma^{-4}$.

Changing back from $\Phi_L(\xi)$ to $\bar{\Phi}(\xi)$ of (7.15) adds phases of $O(\xi^{-3/8})$ (cf. (7.6)) and of $O(\gamma\xi^{-1/4})$; the latter contribution appears in the transformation removing the terms proportional to γ in (7.15). Both these phases are smaller than the error estimate in (7.32).

We summarize the foregoing in the form of:

<u>Lemma 7.2</u>: If τ_o is sufficiently large, the solutions $u(\tau)$ of eqn. (7.6) are given on $[\tau_o, C_\varepsilon \varepsilon^{-3/8}]$ by:

$$u(\tau) = [\eta(\tau)]^{-1/2} R(\xi;\varepsilon) \cos(\xi + \Phi(\xi)) \tag{7.35}$$

with $\eta(\tau)$ of (7.5), ξ of (7.8), $R(\xi;\varepsilon)$ of (7.31) and $\Phi(\xi) = \Phi_L(\xi)$ of (7.32). The solutions depend on two parameters $R_{oL}(\varepsilon), \Psi_{oL}(\varepsilon)$ (cf. eqn.(7.33)).

From Lemma 7.2 it follows that, at $t \simeq C_\varepsilon$ (cf. eqn. (7.4))

$$v(C_\varepsilon) \simeq R_{oL}(\varepsilon)\, \varepsilon^{3/16}\, C_\varepsilon^{-1/6} \exp[-C_\varepsilon \mu/\varepsilon] \tag{7.36}$$

Clearly, $v(C_\varepsilon) \to 0$ as $\varepsilon \to 0$, since $R_{oL}(\varepsilon)$ is bounded. The derivative $du/d\tau$ is obtained from (7.35) using (7.15) for $dR/d\xi$, $d\Phi/d\xi$:

$$\frac{du}{d\tau} = -\sqrt{3}\, R_{oL}\, e^{-s(\xi)}\, \eta^{1/2}\left[\sin\left(\xi + \phi(\xi)\right) + O(\gamma\tau^{-1/3}, \tau^{-1/2})\right] \tag{7.37}$$

Thus, at $t \simeq C_\varepsilon$:

$$\frac{dv}{dt}(C_\varepsilon) \sim \varepsilon^{-5/16}\, C_\varepsilon^{1/6}\, \exp\left(-\mu\, C_\varepsilon/\varepsilon\right) \tag{7.38}$$

Notice, if $\varepsilon/\mu \to 0$ ($\Delta \to \infty$) as $\varepsilon \to 0$, but not quickly enough (e.g. $\mu \sim k\varepsilon \ln \frac{1}{\varepsilon}$), $dv/dt(C_\varepsilon) \to \infty$ as $\varepsilon \to 0$, for any C_ε obeying (7.7).

We obtain next the behaviour of v(t), eqn. (7.2), for t on $[C_\varepsilon, \pi/2]$. Changing variables to:

$$\sigma = \int_{aC_\varepsilon}^{t} X_R(t')\sqrt{3}\, dt' \quad , \quad v(t) = \frac{w(t)}{X_R(t)^{1/2}} \exp\left[-\mu(t - C_\varepsilon)/\varepsilon\right] \tag{7.39}$$

we obtain:

$$\varepsilon \frac{d^2 w}{d\sigma^2} + w\left[1 + \varepsilon R(3X_R^2) - \frac{\mu^2}{3\varepsilon}\frac{1}{X_R^2}\right] + \frac{w^2}{X_R^{3/2}} \exp\left[-\frac{\mu}{\varepsilon}(t - C_\varepsilon)\right] + \frac{w^3}{3X_R^3}\exp\left[-2\frac{\mu}{\varepsilon}(t - C_\varepsilon)\right] = 0 \tag{7.40}$$

where $R(\phi)$ is given by (3.21) and we are interested in the solutions of (7.40) which obey at $t = C_\varepsilon$ initial conditions derived from (7.36), (7.38). The solution we need is obtained in a standard manner by transforming (7.40) to an integral equation using the solutions $\cos(\sigma\,\varepsilon^{-1/2})$, $\sin(\sigma\,\varepsilon^{-1/2})$ of the (incomplete) linear part of (7.40). The free term of the equation is:

$$w^{(o)}(t) = \varepsilon^{3/16}\, R_{oL}(\varepsilon)\cos\left(\xi_R + \phi(\xi_R) + \sigma\varepsilon^{-1/2}\right) e^{-C_\varepsilon \mu/\varepsilon} \tag{7.41}$$

However, it is not true that $w^{(o)}(t)$ is a good approximation to v(t) on $[C_\varepsilon, \pi/2]$, unless :

$$\mu/(\varepsilon \ln 1/\varepsilon) \longrightarrow \infty \qquad (\Delta/\ln\Gamma \to \infty) \text{ as } \varepsilon \to 0 \tag{7.42}$$

Indeed, one verifies that a condition for the mapping $w - w^{(o)} \equiv r \to r^{(o)} + Ar$, given by the integral equation to be contractive is:

$$C_\varepsilon^{-1/2}\, \varepsilon^{-5/16} \exp\left[-2\, C_\varepsilon\, \mu/\varepsilon\right] < 1 \tag{7.43}$$

Since $C_\varepsilon \to 0$ as slowly as we wish as $\varepsilon \to 0$, we conclude that (7.42) is indeed sufficient.

Let now $v^{(o)}(t)$ correspond to $w^{(o)}(t)$ by (7.39). We may state then:

<u>Lemma 7.3</u> : If (7.42) holds, then:

$$|v(t) - v^{(o)}(t)| = \frac{\exp[-\mu t/\varepsilon]}{X_R(t)^{1/2}}\, O(\varepsilon^{-1/8}\exp(-C_\varepsilon\,\mu/\varepsilon)) \tag{7.44}$$

uniformly on $[C_\varepsilon, \pi/2]$.

This closes the description of the special solution $v(t)$ of (7.2).

VIII. Periodic solutions if $\ln \Gamma \prec \Delta \prec \Gamma^{1/4}$

In this Section, we apply Newton's method (Sect.III) to improve $x_a(t)$, eqn. (7.1) to an odd periodic solution of Duffing's equation and then prove its uniqueness (as described in Sect. II).

First, $x_a(t)$ is not itself odd periodic for two reasons: (i) the solution $x_R(t)$ differs from the outer expansion $x_o^{(K,L)}(t)$ (cf. eqn.(4.41)) for $t > a > 0$ uniformly by quantities of $O(\varepsilon^{cP(S)})$ (cf. Lemma 6.5), with $P(S)$ of eq.(4.42), and $c > 0$; (ii) the additional term $v(t)$ is such that $v(\pi/2) \neq 0$, $\dot{v}(\pi/2) \neq 0$. We modify then $x_a(t)$ to:

$$\tilde{x}_a(t) = x_a(t;\varepsilon) + h(t;\varepsilon) \tag{8.1}$$

where $h(t;\varepsilon)$ is C^2, chosen so as to render $x_a(t)$ odd periodic and, for convenience, such that $h(t) = 0$ for $t \in [-\pi/2, a]$, $0 < a < \pi/2$:

$$h(t) = \lambda(t)(x_o^{(K,L)}(t) - x_a(t)) \tag{8.2}$$

with $\lambda(t)$ of class C^2, equal to unity on $[b, \pi/2]$, $a < b$. The residual of $\tilde{x}_a(t)$ is:

$$\mathcal{Y}(\tilde{x}_a)(t) = \mathcal{Y}_o(h)(t) + 3\, x_a^2(t)\, h + 3\, x_a(t)\, h^2 \tag{8.3}$$

since $\mathcal{Y}(x_a)(t) = 0$. Using (7.41), (7.44) and Lemma 6.5 we obtain:

$$\| \mathcal{Y}(\tilde{x}_a) \| = O(\varepsilon^{cP(S)}, \varepsilon^{3/16} \exp[-\mu a/\varepsilon]) \tag{8.4}$$

where $\mathcal{Y}(\tilde{x}_a)(t) = 0$ for $t < a$.

We estimate next the solutions of the variational equation to (1.3) around (8.1):

$$\varepsilon \ddot{x} + 2\mu \dot{x} + 3\tilde{x}_a^2(t)\, x = 0 \tag{8.5}$$

For $|t| < C_\varepsilon$, we use (7.4), (7.5) and obtain:

$$\frac{d^2x}{d\tau^2} + 2\Upsilon \frac{dx}{d} + 3\eta_e^2(\tau)\, x = 0 \tag{8.6}$$

with:

$$\eta_e(\tau) = (x_R(\tau) + v(\tau))/\varepsilon^{1/8} \tag{8.7}$$

Concerning (8.6), we have:

Lemma 8.1: Eqn. (8.6) admits of two linearly independent solutions which for $\tau \in I_\tau(\varepsilon)$ and τ_o sufficiently large, may be written as:

$$u_1(\tau) = \frac{R(\xi)}{[\gamma_e(\tau)]^{1/2}} \sin(\xi + \phi(\xi))\left[\frac{7}{12} R_{Lo}(\varepsilon) J(\Gamma;\xi;3/4) + O(\tau^{-1/6}, \Gamma^{1/2})\right]$$

$$+ e^{-s(\xi)} \frac{1}{[\gamma_e(\tau)]^{1/2}} \cos(\xi + \phi(\xi))(1 + O(\tau^{-1/6}, \Gamma^{1/2}) \equiv \bar{u}_1(\tau) e^{-\Gamma\tau} \tag{8.8}$$

$$u_2(\tau) = \frac{R(\xi)}{[\gamma_e(\tau)]^{1/2}} \sin(\xi + \phi(\xi))(1 + O(\tau^{-1/6}, \Gamma^{1/2}) - e^{-s(\xi)} \frac{1}{[\gamma_e(\tau)]^{1/2}}$$

$$\cos(\xi + \phi(\xi))\, O(\Gamma^{1/2}, \tau^{-1/6}) \equiv \bar{u}_2\, e^{-\Gamma\tau} \tag{8.9}$$

Proof: At a given ε, the solution $u(\tau)$ of (7.6) is fixed by two parameters, $R_{Lo}(\varepsilon)$, $\bar{\phi}_{Lo}(\varepsilon)$ which depend in turn on the initial conditions. The derivatives $\partial u/\partial R_{Lo}$, $\partial u/\partial \phi_{Lo}$ satisfy the variational equation. Using the chain rule, e.g.

$$u_1 \equiv \partial u_1/\partial R_{Lo} = (\partial u/\partial R)(\partial R/\partial R_{Lo}) + (\partial u/\partial \phi)(\partial \phi/\partial R_{Lo}) \tag{8.10}$$

we obtain from (7.35) eqns. (8.8-9), provided we show that $\partial R/\partial R_{Lo}$, $\partial \phi/\partial R_{Lo}$, etc. obey the estimates shown there.

Now, using (7.23) and taking derivatives with respect to R_{Lo}, we obtain from (7.21-22) a set of linear equations for $R_{L,R}$, $\phi_{L,R} \equiv \partial R_L/\partial R_{Lo}, \partial\phi_L/\partial R_{Lo}$:

$$\frac{d}{d\xi} R_{L,R} = -\frac{\Gamma}{\xi^{1/4}} R_{L,R}\, h_o(\xi) + a_{1R}(\xi;R;\phi)\, R_{L,R} + a_{1\phi}(\xi;R;\phi)\phi_{L,R} \tag{8.11}$$

$$\frac{d}{d\xi}\phi_{L,R} = -\frac{7}{12} R_L\, R_{L,R} \frac{h_2(\xi)}{\xi^{3/4}} + b_{1R}(\xi;R;\phi)\, R_{L,R} + b_{1\phi}(\xi;R;\phi)\phi_{L,R} \tag{8.12}$$

where $a_{1R} = O(\Gamma R \xi^{-5/8}, \Gamma^2 \xi^{-1/2}, \xi^{-11/8})$, $a_{1\phi} = O(Ra_{1R})$, $b_{1R} = O(R\xi^{-9/8}, \Gamma R/\xi, \xi^{-11/8})$, $b_1 = O(Rb_{1R})$. Using now:

$$F_{L,R}(\xi) = R_{L,R}(\xi)\, e^{\Gamma\tau(\xi)} = R_{Lo}(\varepsilon)\, e^{\Gamma\tau} + \rho_1(\xi) \tag{8.13}$$

$$\phi_{L,R}(\xi) = \Psi_1(\xi) - \frac{7}{12}\int_{\xi_R}^{\xi} R_L(\xi')\, R_{Lo}(\varepsilon)\, h_2(\xi')(\xi')^{-3/4}\, e^{-s(\xi')}\, d\xi' \tag{8.14}$$

we obtain a set of linear inhomogeneous equations for $\rho_1(\xi)$, $\Psi_1(\xi)$, with initial conditions $\rho_1(\xi_R) = \Psi_1(\xi_R) = 0$; both the coefficients and the inhomogeneity have the order of magnitude $O(a_{1R}, a_{1\phi}, b_{1R}, b_{1\phi})$. Applying now Gronwall's Lemma to the set of eqns. for $\rho_1(\ )$, $\Psi_1(\ )$, we obtain:

$$\max(\rho_1(\xi), \Psi_1(\xi)) = O(\Gamma^{1/2}, \tau^{-1/6}) \tag{8.15}$$

which justifies (8.8). Similar changes of variable ($F_{L,\phi}(\xi) \equiv \rho_2(\xi)$, $\phi_{L,\phi}(\xi) = 1 + \Psi_2(\xi)$) lead to the same estimate (8.15) for ρ_2, Ψ_2 and thus to (8.9).

The two functions (8.8-9) are linearly independent ($W(u_1,u_2) \sim R_{oL}\exp(-2\gamma\tau)$). This ends the proof.

Eqn. (8.8) has a remarkable feature: the function $J(\gamma;\xi;3/4)$ increases in an interval of $O(1/\gamma)$ in τ to the value $\gamma^{-1/3}$ and then levels off. This behaviour influences the Floquet exponents (see below).

The solution (8.8-9) of the variational equation (8.6) allows us to be more explicit about the dependence on ε of the two constants $R_{Lo}(\varepsilon)$, $\Phi_{Lo}(\varepsilon)$ appearing in the solution of (7.35). We define:

$$\Psi(\xi;\varepsilon) = \Phi(\xi;\varepsilon) + \frac{7}{24} J(\gamma;\xi;3/4)\, R^2_{oL}(\varepsilon) \tag{8.16}$$

$$\Psi(\varepsilon) = \Psi(\xi_R;\varepsilon) \tag{8.17}$$

and confine ourselves to the situation of small damping, $\mu \prec \varepsilon^{3/4}$ (For the complementary situation, we have to modify the definition of $\Psi(\varepsilon)$, cf. comments following (7.34)). The quantity $\Psi(\varepsilon)$ is different from $\Psi_{oL}(\varepsilon)$, eqn. (7.33), through quantities that have zero limit as $\varepsilon \to 0$. We may then state:

<u>Lemma 8.2</u>: If $\mu \prec \varepsilon^{3/4}$, $R_{oL}(\varepsilon)$, $\Psi(\varepsilon)$ are continuous functions of ε at $\varepsilon = 0$.

The proof of Lemma 8.2 is deferred to Appendix B. We recall that Lemma 7.1 establishes only the boundedness of $R_{oL}(\varepsilon)$, but states nothing about its behaviour as a function of ε. We use Lemma 8.2 below in estimating Floquet exponents. We call $R_o \equiv R_{oL}(\varepsilon=0)$ and similarly Ψ_o. We point out that the limits R_o, Ψ_o are independent of the law $\mu \sim \varepsilon^\beta$ (or $\Delta \sim \Gamma^\alpha$). They are in this sense, "universal" in the domain $\varepsilon \ln \frac{1}{\varepsilon} \prec \mu \prec \varepsilon^{5/8}$. The way in which they are approached is, however, γ dependent.

Finally, for the interval $[C_\varepsilon, \pi/2]$, we have:

<u>Lemma 8.3</u>: If the damping obeys (7.42), the variational equation (8.5) has on $[C_\varepsilon, \pi/2]$ two linearly independent solutions given by:

$$x^{(w)}_{1,2}(t) = \tilde{x}_a(t)^{-1/2} \exp\left(-\frac{\mu}{\varepsilon} t\right)\left[\begin{pmatrix}\cos\\ \sin\end{pmatrix}(\sigma\varepsilon^{-1/2}) + o(1)\right] = \bar{x}^{(w)}_{1,2}\, e^{-\frac{\mu}{\varepsilon}t} \tag{8.18}$$

with σ of eqn. (7.39) and $o(1)$ is uniform on $[C_\varepsilon, \pi/2]$.

<u>Proof</u>: Clearly, we may write: (cf. eqn. (8.2))

$$\tilde{x}_a(t) = x_o^{(K,L)}(t) + v_1(t) \tag{8.19}$$

with $v_1(t)$ supported on $[C_\varepsilon, b]$, and of class C^2 ($b<\pi/2$). Changing in (8.5) variables to:

$$\sigma' = \int_{C_\varepsilon}^{t} \sqrt{3}\, x_o^{(K,L)}(t')dt' \quad , \; x(t) = p(t)\, x_o^{(K,L)^{-1/2}} \exp\left[-\frac{\mu}{\varepsilon}(t - C_\varepsilon)\right] \quad (8.20)$$

we obtain:

$$\varepsilon \frac{d^2p}{d\sigma'^2} + p\left[1 + \frac{2v_1}{x_o^{(K,L)}} + \frac{v_1^2}{\left(x_o^{(K,L)}\right)^2} + \varepsilon R\left[3x_o^{(K,L)^2}\right] - \frac{\mu^2}{3\varepsilon}(x_o^{(K,L)})^{-2}\right] = 0 \quad (8.21)$$

with $R(\Phi)$ of (3.21). Transforming to an integral equation, it is straightforward to show that, if μ obeys (7.42), the departure of the solution of the latter from $a \cos(\sigma' \varepsilon^{-1/2} + \varphi)$ is $o(1)$. on $[C_\varepsilon, \pi/2]$ (if a, φ are chosen to match the initial conditions at $t = C_\varepsilon$). Replacing σ' with σ brings corrections of $O(\varepsilon^{cP(S)})$, with $P(S)$ of (4.42) (cf. Lemma 6.5)).This ends the proof.

We combine now Lemmas 8.1, 8.3 to write out an approximation to a solution of (8.5) on $[\tau_o \varepsilon^{3/8}, \pi/2]$ as:

$$x_{R1,2}^{(a)} = [\chi_i(t;C_\varepsilon)\, \bar{u}_{1,2}(t;\varepsilon) + \chi_{oR}(t;C_\varepsilon)\, \tilde{x}_{1,2}^{(w)}(t;\varepsilon)] \exp\left(-\frac{\mu}{\varepsilon} t\right) \quad (8.22)$$

where $\chi_i(t;C_\varepsilon), \chi_{oR}(t;C_\varepsilon)$ have the same meaning as in (5.42); in (8.22), the $\tilde{x}_{1,2}^{(w)}(t;\varepsilon)$ are linear combinations of the $\bar{x}_i^{(w)}$ in (8.18). We define:

$$\delta(\varepsilon) = J(\gamma; \xi_R; 3/4) - \gamma^{-1/3} \int_0^\infty \frac{e^{-u}}{u^{2/3}} du \quad (8.23)$$

which has the property that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, let

$$I_o = \int_0^\infty \frac{e^{-u}}{u^{2/3}} du = \Gamma(1/3) \quad (8.24)$$

and

$$\Psi_1(\varepsilon) = \Psi(\varepsilon) - \frac{7}{24}\delta(\varepsilon) + \frac{3\sqrt{3}}{4} \frac{(aC_\varepsilon)^{4/3}}{\sqrt{\varepsilon}} \quad (8.25)$$

With this, we choose in (8.22):

$$\tilde{x}_1^{(w)}(t;\varepsilon) = \frac{\varepsilon^{1/16}}{\tilde{x}_a^{1/2}} \Big\{ \sin\left[\frac{\sigma}{\sqrt{\varepsilon}} + \Psi_1(\varepsilon) - \frac{7}{24} R_o^2 I_o \gamma^{-1/3}\right] \cdot \frac{7}{12} R_o^2 I_o \gamma^{-1/3} + \cos\left[\frac{\sigma}{\sqrt{\varepsilon}} + \Psi_1(\varepsilon) - \frac{7}{24} R_o^2 I_o \gamma^{-1/3}\right] \quad (8.26)$$

$$\tilde{x}_2^{(w)}(t;\varepsilon) = \frac{\varepsilon^{1/16}}{\tilde{x}_a^{1/2}} \sin\left[\frac{\sigma}{\sqrt{\varepsilon}} + \Psi_1(\varepsilon) - \frac{7}{24} R_o^2 I_o \gamma^{-1/3}\right] \quad (8.27)$$

The same arguments as in Lemma 5.3 allow us to state:

<u>Lemma 8.4</u>: If the damping obeys (7.42), eqn. (8.5) admits on $[\tau_o \varepsilon^{3/8}, \pi/2]$ of two linearly independent solutions of the form:

$$x_{R1,2}(t;\varepsilon) = \left[x_{R1,2}^{(a)}(t;\varepsilon) + o(\varepsilon^{1/16})\right] \exp\left(-\frac{\mu}{\varepsilon} t\right) \equiv x_{R1,2}^{(o)}(t;\varepsilon) \exp\left(-\frac{\mu}{\varepsilon} t\right) \quad (8.28)$$

The proof is the same as in Lemma 5.3. Finally, we have:

Lemma 8.5: If (7.42) holds, eqn. (8.5) admits on $[-\pi/2, \pi/2]$ of two linearly independent solutions of the form: (k=1,2)

$$x_k(t) = \Big[\chi_{oL}(t;\varepsilon^{\alpha})\, u_{koL}(t;\varepsilon) + \chi_i(t;\varepsilon)u_{ki}(\tau;\varepsilon) + \chi_{oR}(t;C_\varepsilon)\, u_{koR}(t;\varepsilon) + o(\varepsilon^{1/16})\Big] \exp\Big[-\frac{\mu}{\varepsilon}(t+\pi/2)\Big] \equiv \bar{x}_k(t)\, \exp\Big[-\frac{\mu}{\varepsilon}(t+\pi/2)\Big] \qquad (8.29)$$

where: $u_{koL}(t;\varepsilon) = u_{ko}(t;\varepsilon)$ of eqn. (5.19), $u_{ki}(\tau;\varepsilon)$ are the same as in (5.18) for $\tau < 0$; the solutions $u_{koR}(t;\varepsilon)$ may be written as:

$$u_{koR}(t;\varepsilon) = \sum_{j=1}^{2} a_{kj}(\varepsilon)\, x_{Rj}(t;\varepsilon) \qquad (8.30)$$

where the $x_{Rj}(t;\varepsilon)$ are the same as in (8.28). The quantities $a_{kj}(\varepsilon)$ have limits as $\varepsilon\to 0$ and $\det|a_{ij}(\varepsilon)| = 1$.

The proof is the same as that of Lemma 5.4;it uses the continuity at $\varepsilon = 0$ of $R_{Lo}(\varepsilon), \Psi_{Lo}(\varepsilon)$, established in Lemma 3.2. We omit the details.

We have now all the requisites for the proof of existence and uniqueness of periodic solutions, if the damping obeys (7.42). To improve $x_a(t)$, eqn.(8.1) to $x_p(t)$ by Newton's method, we need (8.4) and:

Lemma 8.6: If $\varepsilon \ln(1/\varepsilon) \prec \mu \prec \varepsilon^{5/8}$

$$\| [\mathcal{D}_x(x_a)]^{-1} \| = o(\gamma^{-1/3}\, \varepsilon^{-5/8}) \qquad (8.31)$$

The proof is immediate, using the fact that $x_k(t)$, eqn. (8.29) are bounded absolutely by $\gamma^{-1/6}$ (this value is attained at $\tau \sim 1/\gamma$,cf. eqn.(8.8)).

With this, as in Sect. III, eqns.(3.8), it follows that the conditions for a ball of radius ε^q in $\mathbb{D}(-\pi/2, \pi/2)$ around $\breve{x}_a$ to be mapped into itself are (assuming the second term is dominant in (8.4)):

$$\gamma^{-1/3}\, \varepsilon^{-5/8} \max\Big[\varepsilon^{cP}, \varepsilon^{3/16} \exp -\frac{\mu}{\varepsilon} a\Big] \prec \varepsilon^q \;,\; \gamma^{-1/3}\, \varepsilon^{-5/8}\, \varepsilon^q \Big] \prec 1 \qquad (8.32)$$

If (7.42) is obeyed, these conditions are satisfied, in fact for any $q > 0$, provided the integer P in (8.4) is appropriately large and ε appropriately small. Eqns. (8.32) also ensure contractivity of the mapping in eqn.(3.8). Thus the existence of $x_p(t)$ and its approximation by $\breve{x}_a(t)$ within ε^q is established.

The procedure of proving uniqueness outlined in Sect. II may be applied directly to our situation. Choose $\lambda = \lambda_o$ (independent of ε) $< \mu/\varepsilon - (1/3\pi)\ln\gamma$. As in Sect. VI, we verify $K_B(\varepsilon;\lambda) = C_1$, $K_D(\varepsilon;\lambda) = C_2\varepsilon^{-3/8}$, $g(\varepsilon) = \varepsilon^{1/2}$, $T(\varepsilon) = 4$ (cf. eqn. (2.36)), $S(\varepsilon) = \varepsilon^{5/8}$. According to (2.33), if:

$$|u(t_o)| + \sqrt{\varepsilon}\,|\dot{u}(t_o)| < M < \varepsilon^{5/8} \tag{8.33}$$

and if $4\exp(-\lambda_o\pi) < 1$, the solutions $u(t)$ of eqn.(2.12) starting at $t=t_o$ in $\mathcal{U}(M)$ (cf. eqn. (2.25)) tend to zero as $t\to\infty$. For ε small enough, λ_o may be appropriately chosen and Lemma 2.2 shows that, in each half period of the external force, any solution $x(t)$ lying in the domain D of (2.1) approaches $x_p(t)$ so that $u(t) = x(t) - x_p(t)$ fulfills (8.33), for some t_o. This establishes the uniqueness of $x_p(t)$.

We have thus proved:

Theorem 8.1: If $\ln\Gamma \prec \Delta \prec \Gamma^{1/4}$, eqn. (1.3) admits for Γ large enough of a unique periodic solution $x_p(t)$; it is uniformly approximated on $[-\pi/2, \pi/2]$ by $\tilde{x}_a(t;\varepsilon)$, eqn. (8.1), together with its first two derivatives, as well as one wishes, if Γ is appropriately large.

Finally, we compute the Floquet multipliers of $x_p(t)$, using eqns.(2.38),(2.40) and Lemma 8.5. We write (k=1,2):

$$u_{koR}(t;\varepsilon) = \varepsilon^{1/16}\, c_k(\varepsilon)\, \tilde{x}_a^{-1/2}\, \sin[\chi(\varepsilon) + \Theta_k(\varepsilon)] \tag{8.34}$$

where

$$c_k(\varepsilon) = \frac{7}{12}\, R_{oL}^2(\varepsilon)\, I_o\, \gamma^{-1/3}\, a_{k1}(\varepsilon)\, (1 + O(\gamma^{1/3})) \tag{8.35}$$

(if $a_{11}, a_{21} \neq 0$, at least one of them obeys this), and (cf.(8.26),(8.27))

$$\chi(\varepsilon;t) = \varepsilon^{-1/2}\int_{C_\varepsilon}^{t} [x_p(t')]^{1/2}\sqrt{3}\,dt' + \psi_1(\varepsilon) - \frac{7}{24} R_{oL}^2(\varepsilon) I_o \gamma^{-1/3} \tag{8.36}$$

It is true that, independently of the values of a_{11}, a_{21} (cf. eqn. (5.48))

$$c_1(\varepsilon)\, c_2(\varepsilon)\, \sin(\Theta_1(\varepsilon) - \Theta_2(\varepsilon)) = 1 \tag{8.37}$$

which implies that, if $a_{11}(\varepsilon) \neq 0$, $a_{21}(\varepsilon) \neq 0$, $\Theta_1(\varepsilon) - \Theta_2(\varepsilon) = O(R_{oL}^{-4}(\varepsilon)\, I_o^{-2}\, \gamma^{2/3})$. It turns then out that (i=1,2), using

$$\psi_L = \int_{-\pi/2}^{-b\varepsilon^\alpha} \varepsilon^{-1/2}\sqrt{3}\,|x_p(t')|^{1/2}\,dt' \quad : \tag{8.38}$$

$$f_{i1} = c_i \sin[\chi(\varepsilon;\pi/2) + \theta_i + \psi_L], \quad f_{i2} = c_i \cos[\chi(\varepsilon;\pi/2) + \theta_i + \psi_L] \tag{8.39}$$

so that:

$$\mathrm{Tr}\,\tilde{F} = R_{oL}^2(\varepsilon)\, I_o\, \gamma^{-1/3} \sin\left[\chi(\varepsilon;\ /2) + \frac{\theta_1+\theta_2}{2} + \psi_L + \beta(\varepsilon)\right] d(\varepsilon) \tag{8.40}$$

where $\beta(\varepsilon)$, $d(\varepsilon)$ depend on the $a_{ij}(\varepsilon)$. With (2.40):

$$\tilde{\mu}_{1,2}(\varepsilon) = e^{-\frac{\mu}{\varepsilon}\pi}\left[R_{oL}^2(\varepsilon)\, I_o\, \gamma^{-1/3}\left(d(\varepsilon)\sin\Psi(\varepsilon) \pm \left(d^2(\varepsilon)\sin^2\Psi(\varepsilon) - \gamma^{2/3}/R_o^4\, I_o^2\right)^{1/2}\right)\right] \tag{8.41}$$

where

$$\Psi(\varepsilon) = \psi_L(\varepsilon) + \psi_R(\varepsilon) + \psi_1(\varepsilon) - \frac{7}{24} R^2_{oL}(\varepsilon)\, I_o \gamma^{-1/3} + \frac{\theta_1 + \theta_2}{2} + \beta(\varepsilon) \tag{8.42}$$

Eqns. (8.41-42) display the effect mentioned in the introduction (cf. eqn. (1.17)): as a function of ε, $\Psi(\varepsilon)$ increases without bounds and passes the value $k\pi$ in approximately equidistant points in the variable $\varepsilon^{-1/2} = \Gamma^{1/3}$; the distance in $\Gamma^{1/3}$ between two successive passages through zero (mod π) is given by (1.16). As $\varepsilon \to 0$, $d(\varepsilon)$ has a nonzero limit, and thus there are intervals where the multipliers in (8.41) are real. The maxima of $\tilde{\mu}_{1,2}(\varepsilon)$ are approximately equidistant (better as $\varepsilon \to 0$), with a spacing given by (1.16) and a magnitude equal to:

$$\tilde{\mu}_{max}(\varepsilon) = \exp\left(-\frac{\mu}{\varepsilon}\pi\right) \frac{2d\, R_o^2\, I_o}{\gamma^{1/3}} \tag{8.43}$$

Eqn. (8.43) shows that $\tilde{\mu}_{max}(\varepsilon)$ becomes of the order unity if

$$\mu \sim \frac{1}{8\pi}\, \varepsilon \ln \frac{1}{\varepsilon} - \frac{\varepsilon}{3} \ln \ln \frac{1}{\varepsilon} + \ldots \tag{8.44}$$

i.e.

$$\Delta \sim \frac{1}{12\pi} \ln \Gamma \tag{8.45}$$

Clearly, the domain of validity of (8.43) is, strictly speaking, given by (7.42); there, $\tilde{\mu}_{max}(\varepsilon) \to 0$ as $\varepsilon \to 0$. We do not expect the simple law (8.42) to retain its validity if $\tilde{\mu}_{max}(\varepsilon) \sim 1$. Indeed, the bifurcations occurring from $x_p(t)$ are of different types according to whether $\tilde{\mu}_{max}(\ \ ) = +1$ or -1; the former case leads to saddle node bifurcations, through which $x_p(t;\varepsilon)$ disappears; the latter to flip bifurcations (appearance of stable periodic, but not odd periodic solutions). The curves $\tilde{\mu}(\varepsilon;\mu) = +1$ are expected to meet at a cusp in the Γ, Δ plane, which is not true for $\tilde{\mu}(\varepsilon;\mu) = -1$. However, in eqn. (8.44), the situations $\tilde{\mu} = \pm 1$ are completely symmetrical. Thus, although it is too simple to describe bifurcations, eqn. (8.43) makes it plausible that the loss of uniqueness and stability of $x_p(t)$ occurs in the domain $\Delta \sim \text{const} \ln \Gamma$ of the parameter plane.

This closes our discussion of periodic solutions of Duffing's equation.

IX. Some final remarks:

We have presented a detailed analysis of the periodic solutions of Duffing's equation (1.1) at large forcing and damping. Depending on the way in which the damping Δ increases with the outer force Γ, the appearance of the periodic solutions varies; we have derived above their asymptotic expansions in those domains of parameters where their uniqueness can also be demonstrated: $\Delta / \ln \Gamma \to \infty$ as $\Gamma \to \infty$. With slight more care in our arguments, it is plausible that we can extend the domain of uniqueness down to the asymptotic "line" $\Delta \sim k \ln \Gamma$, for some large k.

If $\Delta \prec \Gamma^{2/3}$, the periodic solutions of (1.3) are close to $(\sin t)^{1/3}$. The difference of the periodic solution $x_p(t;\Gamma)$ to $(\sin t)^{1/3}$ may also be examined in their (corresponding) Fourier coefficients. The coefficients of $(\sin t)^{1/3}$ fall off like $n^{-4/3}$; those of the periodic solution drop off exponentially, since $x_p(t;\Gamma)$ is holomorphic in $\zeta = \exp(it)$ in a ring around $|\zeta| = 1$. The first coefficients of $x_p(t;\Gamma)$ and of $(\sin t)^{1/3}$ are almost identical; the difference in the high coefficients is due to boundary layer corrections. As an example, at small damping ($\Delta \prec \Gamma^{1/4}$), $x_p(t;\Gamma)$ oscillates with increasing frequency ω as $\Gamma \to \infty$ in a time of $O(1/\Delta)$ to the right of $t = 0$; we expect thus a structure in the spectrum of the high Fourier coefficients, which moves to higher ω as $\Gamma \to \infty$. We can be more explicit about this motion, in a rough approximation: the contribution to the Fourier spectrum of the oscillations may be written: ($\lambda = \omega\, \varepsilon^{3/8}$) (cf.eq.(7.35))

$$F(\omega;\varepsilon) \simeq \int_{\tau_0 \varepsilon^{3/8}}^{1/\Delta} t^{-1/6} \exp(i\omega t + i t^{4/3} \omega_0 \varepsilon^{-1/2})\,dt \approx \varepsilon^{5/16} \int_{\tau_0}^{1/\gamma} \exp\left(-\gamma\tau + i\lambda\tau + i\tau^{4/3}\omega_0\right) \tau^{-1/6}\, d\tau \tag{9.1}$$

As $\gamma \to 0$, the last integral is convergent, so that $F(\omega;\varepsilon) \sim \varepsilon^{5/16} f(\lambda)$ in this limit. This shows there is a structure in the spectrum which decreases in magnitude staying selfsimilar and moves towards frequencies increasing like $\Gamma^{1/4}$. This is one example of many scaling laws of $x_p(t;\Gamma)$ which have their origin in the boundary layer structure of (1.3) (see Ref.[7] for a more detailed discussion).

Clearly, the reason why boundary layers appear is that no harmonic term is present in (1.1). If a positive term existed, increasing like $\Gamma^{2/3}$, then, as $\Gamma \to \infty$, eqn. (1.3) would reduce to

$$cx + x^3 = \sin t \tag{9.2}$$

The solution $x_{oo}(t)$ of (9.2) has bounded derivatives of all orders, for all t and the periodic solution of Duffing's equation is obtained by direct iteration, similarly to the outer expansion.

If an harmonic term independent of Γ is present, then the limiting solution of (the analogon of) (1.3) is again $(\sin t)^{1/3}$. In this equation, the harmonic term is $O(\varepsilon)$ and does not manifest itself in the first two terms of the outer asymptotic expansion. In the inner expansion, the harmonic term is of the second order in the appropriate small parameters, if $\Delta \succ \Gamma^{1/4}$ (it is $O(\mu^{6/5}(\varepsilon/\mu^{8/5}))$, otherwise of $O(\varepsilon^{3/4})$. Thus, the analysis of this paper may be repeated without major changes if an harmonic term, independent of Γ and of either sign is present in eqn. (1.1).

The numerical tasks that are left in the description of periodic solutions of (1.3) are related to the boundary layer equations and to the determination of the transition matrices $a_{ij}(\varepsilon)$, eqns. (8.30), (5.44). Using Lemma 8.2, the two quantities d, R_o appearing in the expressions for the Floquet multipliers, eqn. (8.43), may be evaluated at $\varepsilon = 0$, i.e. we have only to consider eqn. (A.31) and its variational associate. We obtain $d \doteq 0.38$, $R_o \doteq 0.86$.

The expected logarithmic increase with Γ of the curves $\tilde{\mu}_{max}(\Delta, \Gamma) =$ const < 1 is a consequence of the factor $\hat{\gamma}^{1/3}$ in the denominator of (8.43). For small damping, $\gamma \sim \varepsilon^{3/8} \sim \Gamma^{-1/4}$. The values of Γ at which such an increase is practically visible are, however, quite large: one criterion for this is that the $\xi^{1/4}$ increase of the phase of the solution of (A.31) be clearly visible for values of ξ less than $1/\hat{\gamma}^{4/3}$, i.e. before the effect of the damping sets in. Numerical experiments with (A.31) show that, e.g. $\xi \sim 2\text{-}3 \times 10^3$ is appropriate; equating this to $1/\hat{\gamma}^{4/3}$ leads to $\Gamma \sim 10^{10}$.

On closing, we point out that some of the results we obtained (e.g. Section VII) retain their validity even along the line $\Delta \sim \kappa \ln \Gamma$. There is thus some reason to believe that the bifurcations presumably occurring there may be amenable to an accurate treatment. We hope to return to this question in the future.

Acknowledgements: I thank Prof. G. Höhler for several discussions and questions.

Appendix A: Proof of Lemma 7.1.

The proof proceeds via a number of Lemmas that are displayed below. The object of interest is the energy associated to (7.9):

$$E(p;\xi;\varepsilon) = \frac{1}{2}\left(\frac{dp}{d\xi}\right)^2 + \frac{p^2}{2}(1 + K(\xi;\varepsilon)) + \frac{p^3}{3}\,\frac{h_1(\xi;\varepsilon)}{\xi^{3/8}} + \frac{p^4}{12\,\xi^{3/4}}\,h_2(\xi;\varepsilon) \qquad (A.1)$$

which is positive definite for all ξ sufficiently large. It is true that:

$$\frac{dE}{d\xi} = -\frac{2\Upsilon}{\xi^{1/4}}\,h_o(\xi)\left(\frac{dp}{d\xi}\right)^2 + \frac{p^2}{2}\frac{dK}{d\xi} + \frac{p^3}{3}\frac{d}{d\xi}\left(\frac{h_1(\xi)}{\xi^{3/8}}\right) + \frac{p^4}{12}\frac{d}{d\xi}\left(\frac{h_2(\xi)}{\xi^{3/4}}\right) \qquad (A.2)$$

The first term in (A.2) is strictly negative, so that:

$$\frac{dE}{d\xi} < \frac{p^2}{2}\frac{dK}{d\xi} - \frac{p^3}{8}\frac{1}{\xi^{11/8}}\left(h_1(\xi) + O\left(\frac{dh_1}{d\xi}\right)\right) - \frac{p^4}{16}\frac{1}{\xi^{7/4}}\left(h_2(\xi) + O\left(\xi\frac{dh_2}{d\xi}\right)\right) \quad (A.3)$$

With (7.10), the terms in brackets are bounded from above and below on $I_\xi(\varepsilon)$ by positive constants $C_i \pm \Delta_i$, i=1,2, independent of ε, for small ε. Integration of (A.3) does not lead directly to the statement of the Lemma since it is a priori possible that the integral over the cubic term is negative divergent. This occurrence is forbidden if we prove that $|p| <$ const $\xi^{1/3-\delta}$, for some $\delta > 0$. The statements below justify such a bound (Lemmas A.4,A.5), essentially as a consequence of the oscillatory character of the motion.

It is convenient to change variables to:

$$w = p\,\xi^{-3/8} \qquad (A.4)$$

so that:

$$\frac{d^2w}{d\xi^2} + \frac{dw}{d\xi}\left(\frac{3}{4\xi} + \frac{2\Upsilon}{\xi^{1/4}}h_o(\xi)\right) + w(1 + K_1(\xi;\Upsilon)) + w^2\,h_1(\xi;\varepsilon) + \frac{w^3}{3}\,h_2(\xi;\varepsilon) = 0 \quad (A.5)$$

where:

$$K_1(\xi;\Upsilon) = K(\xi;\Upsilon) - \frac{15}{64}\xi^{-2} + \frac{3\Upsilon}{4}\xi^{-5/4}\,h_o(\xi) \qquad (A.6)$$

Eqn. (A.5) has the merit that the coefficients of w^2, w^3 are very close to constants, if $\xi \in I_\xi(\varepsilon)$. The first statement is:

Lemma A.1: The solutions $w(\xi)$ of (A.5) are bounded, uniformly with respect to ε, for $\xi \in I_\xi(\varepsilon)$.

Proof: The energy function of (A.5) is:

$$L(\xi;\varepsilon) = \frac{1}{2}\left(\frac{dw}{d\xi}\right)^2 + w^2\left[\frac{1}{2}(1 + K_1(\xi;\gamma)) + \frac{w}{3}h_1(\xi) + \frac{w^2}{12}h_2(\xi)\right] = \frac{1}{2}\left(\frac{dw}{d\xi}\right)^2 + w^2 H(\xi;w) \tag{A.7}$$

which is positive definite in the $(w, dw/d\xi)$ plane. Also:

$$\frac{dL}{d\xi} = -\left(\frac{dw}{d\xi}\right)^2\left[\frac{3}{4\xi} + \frac{2\gamma}{\xi^{1/4}}h_o(\xi)\right] + \frac{w^2}{2}\frac{dK_1}{d\xi} + \frac{w^3}{3}\frac{dh_1}{d\xi} + \frac{w^4}{12}\frac{dh_2}{d\xi} \tag{A.8}$$

But one verifies easily that $|w|^3$, $|w|^4 <$ const $L(\xi)$, for some constant, independent of ξ. Thus, (A.8) implies:

$$\frac{dL}{d\xi} < \text{const } L(\xi)\left(\frac{1}{2}\frac{dK_1}{d\xi} + \frac{dh_1}{d\xi} + \frac{dh_2}{d\xi}\right) \tag{A.9}$$

Integrating (A.9), we see that $L(\xi)$ is bounded for $\xi \in I_\xi$, which proves Lemma A.1.

We conclude:

$$|p(\xi)| < C\,\xi^{3/8} \tag{A.10}$$

for $\xi \in I_\xi(\varepsilon)$, with C independent of ε (but not on the initial conditions).

The next Lemma states that the solutions $w(\xi)$ of (A.5) are oscillatory, if the energy is sufficiently large. Let to this end:

$$t(\xi) = \max\left[\gamma\xi^{-1/4}, \xi^{-1}\right] \tag{A.11}$$

and $\bar{I}_\xi(\varepsilon;C)$ a subinterval of $I_\xi(\varepsilon)$ where the inequality $Ct^2(\xi) < L(\xi) < L_o$ holds, for a sufficiently large constant C. Then, we have:

Lemma A.2: As long as $\xi \in \bar{I}_\xi(\varepsilon;C)$, there exist constants $C_1, C_2 > 0$, independent of ε (but depending on L_o and C) so that, if ε is small enough:

$$w(\xi_n) = 0$$

at points $\xi_n \in \bar{I}_\xi$, with $C_1 < \xi_{n+1} - \xi_n < C_2$.

Proof: Consider the family of potentials, with $\bar{\xi}$ a fixed parameter:

$$V_o(w;\bar{\xi}) = \frac{w^2}{2} + \frac{w^3}{3}h_1(\bar{\xi};\varepsilon) + \frac{w^4}{12}h_2(\bar{\xi};\varepsilon) \tag{A.12}$$

Using (7.10-11), we verify that, for any $\bar{\xi} \in I_\xi(\varepsilon)$, provided ε is small enough, the motion without damping in potential $V_o(w;\bar{\xi})$ is oscillatory. If the energy is less than L_o, then the period of the motion may be bounded from above (because of the harmonic oscillator part) and from below (because of the bounded energy), uniformly with respect to $\bar{\xi}$. We compare then motions $w(\xi)$, $w_o(\xi)$ of (A.5) and in the potential (A.12), with $\bar{\xi} = \xi^{(o)}$, and without damping, starting with the same initial conditions at $\xi = \xi^{(o)}$. Since both motions are bounded for $\xi \in I_\xi$:

$$\left| \frac{\partial V_o}{\partial w}(w^{(o)};\bar{\xi}) - \frac{\partial V_o}{\partial w}(w;\bar{\xi}) \right| \leq \mathcal{L}(L_o)|w - w_o| \tag{A.13}$$

for some Lipschitz constant $\mathcal{L}(L_o)$, uniform in $\bar{\xi}$. It follows then from Gronwall's inequality that:

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$$\sup(|w(\xi) - w_o(\xi)|, |dw/d\xi - dw_o/d\xi|) < C(L_o) \quad t(\xi^{(o)}) \tag{A.14}$$

over a ξ interval of length $O(1)$, since $t(\xi)$ is monotonically decreasing. With (A.14), we can compare the period of $w(\xi)$ with the distance between two successive passages through zero (in the same sense) of $w(\xi)$, as long as $\xi \in \bar{I}$. Let $w_o(\xi_o') = 0$. Since $w(\xi) = (w - w_o)(\xi) + w_o(\xi)$, the zero of $w(\xi)$ is contained between the two roots of:

$$w_o(\xi) = \pm \sup_{|\xi|} |w - w_o| \tag{A.15}$$

lying on opposite sides of ξ_o'. Assume now $L(\xi^{(o)}) > C\, t^2(\xi^{(o)})$, with, e.g. $C \geq 2\, C^2(L_o)$ of (A.14). Further, we can assume $t(\xi^{(o)})$ is so small that, for $\xi \in \bar{I}$, the bound (A.14) implies: $V_o(|w-w_o|;\bar{\xi}) < C^2(L_o)\; t^2(\xi^{(o)})$. This allows one to bound from below $|dw_o/d\xi|$ over the interval of ξ where the roots of (A.15) lie:

$$\left(\frac{dw_o}{d\xi}\right)^2 > 2\,(L(\xi^{(o)}) - C^2(L_o)\; t^2(\xi^{(o)})) > L(\xi^{(o)}) \tag{A.16}$$

With this, using the mean value theorem over the above interval of ξ:

$$|w_o(\xi)| > \inf |dw/d\xi|\,|\xi - \xi_o'| > L(\xi_o)^{1/2}|\xi - \xi_o'| \tag{A.17}$$

which gives, with (A.15), the desired bound on $\xi - \xi_o'$:

$$|\xi - \xi_o'| < C(L_o)\; L(\xi_o')^{-1/2}\; t(\xi^{(o)}) \tag{A.18}$$

Choosing now the constant C in the definition of $I(\xi;C)$ or $\xi^{(o)}$ sufficiently large, the statement of the Lemma follows.

We call $\{\xi_n\}$ a (possible) sequence of zeros of $w(\xi)$ in the same sense, for $\xi \in \bar{I}(\varepsilon;C)$. With a comparison similar to the one of Lemma B.2, we conclude that the time taken by $dw/d\xi$ to decrease from the value $(2L(\xi_n))^{1/2}$ to, e.g.

$(L(\xi_n)/2)^{1/2}$ is bounded from below by a constant C_1'.

We describe now the change of the energy $L(\xi;\varepsilon;w)$ with ξ :

Lemma A.3: For any $\delta > 0$, there exists $\varepsilon_o > 0$ so that, if $\varepsilon < \varepsilon_o$, $L(\xi;\varepsilon)$ reaches the value δ at $\xi = \xi_1 \subset [\xi_o, \gamma^{-4/3}]$ and stays less than δ for $\xi > \xi_1$, $\xi \in I_\xi(\varepsilon)$.

Proof: Assume $L(\xi) = \delta_1$ at $\xi = \xi_1$. Integrating (A.9) from ξ_1 onwards, we obtain that $L(\xi) < \text{const}\cdot\delta_1$, for $\xi \in I_\xi(\epsilon)$. Thus, it is sufficient to show that, for ε samll enough, $L(\xi)$ reaches any given δ, at $\xi_1 < 1/\gamma$.

For $\xi < \gamma^{-4/3}$, $t(\xi) = 1/\xi$. We choose first ε_o so that $1/\gamma < C/\delta^{1/2}$. If the motion is such that $L(\xi;\varepsilon) < C/\xi^2$, for large ξ, the statement of the Lemma follows. The only possibility left is that, for $\xi_1(\varepsilon) < \xi < \gamma^{-4/3}$, $L(\xi) > C\xi^{-2}$. The motion is then oscillatory (Lemma A.2) and we can estimate in more detail the change in $L(\xi;\varepsilon)$, e.g. between two successive passages through zero, in the same sense. We integrate (A.8) and use the fact that $:|w^2|/2, |w|^3/3, |w|^4/12 < C_2 L(\xi)$, for some C_2, independent of :

$$L(\xi_n) - L(\xi_{n+1}) > \int_{\xi_n}^{\xi_{n+1}} \left(\frac{dw}{d\xi}\right)^2 \left[\frac{3}{4\xi} + \frac{2\gamma}{\xi^{1/4}} h_o(\xi)\right] - C_2 \int_{\xi_n}^{\xi_{n+1}} L(\xi)\left[\frac{1}{2}\frac{dK}{d\xi} + \frac{dh_1}{d\xi} + \frac{dh_2}{d\xi}\right] d\xi$$
$$C_3\, L(\xi_n)(3/(4n) + 2\gamma/n^{1/4}) - C_4\, L(\xi_n)(1/n^2 + \varepsilon^{3/4} n^{1/2}) \qquad \text{(A.19)}$$

The lower bound on the first integral in (A.19) is a consequence of the oscillatory character of the motion. For the second term, we have used (A.9). In both terms, we have used Lemma A.2 to replace ξ_n by n. Eqn. (A.19) implies, for $\xi < \gamma^{-4/3}$

$$L(\xi_{n+1}) < L(\xi_n)/(1 + C_5/n) \qquad \text{(A.20)}$$

for some $C_5 > 0$, independent of ε. Iterating (A.20) over k rotations:

$$L(\xi_{n+k}) < L(\xi_n) \prod_{j=0}^{k-1} (1 + C_5/(n+j))^{-1} < C_6\, L(\xi_n)/k^{\alpha} \qquad \text{(A.21)}$$

for some constants $C_6, \alpha > 0$, independent of ε. Clearly, $L(\xi_{n+k})$ becomes smaller than δ in $(\delta/\text{const})^{1/\alpha}$ steps. It is sufficient to choose ε so that these steps take place in $[\xi_o, 1/\gamma^{4/3}]$. This ends the proof of Lemma A.3.

The following Lemma states that $p(\tau)$ decays exponentially for $\tau > 1/\gamma$. Let

$$E_u(\tau) = \frac{1}{2}\left[\frac{du}{d\tau} + 2\gamma u\right]^2 + \frac{3}{2} u^2 \eta^2 + \eta u^3 + \frac{u^4}{4} \qquad \text{(A.22)}$$

with $\eta(\tau)$ of eqn. (7.5). Then:

Lemma A.4: If $\xi > 1/\gamma^{4/3}$

$$|p(\xi)|, |dp/d\xi| < \frac{\text{const}}{\xi^{1/8}} \left[E_u(1/\gamma)\right]^{1/2} e^{-k\gamma\tau(\xi)} \tag{A.23}$$

for some $k > 0$.

The proof is done by changing $u = v \exp(-k\gamma\tau)$ in eqn. (7.6) and using a Liapunov function $E_v(\tau)$, similar to (A.22) and Lemmas 2.2, 2.3. We use then eqn. (7.8).

The energy equation (A.2) allows us to make the rate of decay of oscillations more precise than in (A.21). For ξ_n in $\bar{I}(\varepsilon;C)$, we write: $(dw/d\xi)(\xi_n) \doteq m_n(\varepsilon)$. It follows then that the maximal value of w at the n^{th} turn is $w_n \doteq m_n(\varepsilon)(1 + O(n^{-3/8}))$ $(\xi_n < \gamma^{-4/3})$. We can then state:

<u>Lemma A.5</u>: $m_n(\varepsilon) < \text{const}/n^{3/8-\delta}$, where δ may be made as small as one wishes if ξ_o is large enough and ε small enough. The constant is independent of ε.

<u>Proof</u>: We multiply both sides of (A.2) by $\exp\left[-(K(\xi) - K(\xi_o))\right]$ and integrate between two successive passages through zero of $p(\xi)$, at ξ_n, ξ_{n+1}:

$$(E(\xi_{n+1}) - E(\xi_n)) \exp -K(\xi_{n+1}) + E(\xi_n)\left[\exp\left(-K(\xi_{n+1})\right) - \exp\left(-K(\xi_n)\right)\right] <$$

$$\exp\left(-K(\xi_n)\right)\int_{\xi_n}^{\xi_{n+1}} \frac{|p^3(\xi)|}{8\xi^{11/8}}\left[h_1(\xi) + O(\xi\frac{dh_1}{d\xi})\right] d\xi < D\frac{m_n^3}{n^{1/4}} \tag{A.24}$$

where we have used Lemma B.2 and $|p(\xi)| < m_n n^{3/8}$ to estimate the last term. Further, using Lemma B.2 and (7.12):

$$|K(\xi_n) - K(\xi_{n+1})| < D_1\left(\frac{1}{n^3} + \frac{\gamma}{n^{9/4}}\right) \tag{A.25}$$

$$\exp|K(\xi_n)| < D_2 \tag{A.26}$$

with D_1, D_2 independent of ε, so that (A.24) may be written as:

$$\frac{1}{2} m_{n+1}^2 (n+1)^{3/4} - \frac{1}{2} m_n^2 n^{3/4} - \frac{D_3}{2} m_n^2 n^{3/4}\left(\frac{1}{n^3} + \frac{\gamma}{n^{9/4}}\right) < D\frac{m_n^3}{n^{1/4}} \tag{A.27}$$

This means:

$$m_{n+1} < m_n\left[1 - \left(\frac{3}{4n} - 2D\frac{m_n}{n} + O(n^{-3}, \gamma n^{-9/4})\right)\left(1 + \frac{m_{n+1}}{m_n}\right)^{-1}\left(1 + \frac{1}{n}\right)^{-3/4}\right] \tag{A.28}$$

But $m_{n+1}/m_n = 1 + O(1/n)$ and Lemma (A.3) implies that $2Dm_n < 3/4$, for n large enough; in fact, by Lemma A.3, the second term is as close to 3/8 as we wish, if we let ε be small enough. Iterating (A.28), we obtain:

$$m_n < \text{const}\, m_{n_o} / n^{3/8-\delta} \tag{A.29}$$

which proves Lemma A.5.

The decay (A.29) means:

$$|p(\xi)| < C\,\xi^{\delta} \tag{A.30}$$

This is (as expected), weaker than $C\xi^{-5/8}$, which is obtained if $L(\xi) < C\,t^2(\xi)$. Multiplying (A.3) by $\exp(-K(\xi))$ and integrating up to $1/\Upsilon^{4/3}$, we obtain $E(1/\Upsilon^{4/3})$ $<$ const, independently of ε (if it is small enough). Further, in eqn. (A.22), $[E_u(1/\delta)]^{1/2} < \Upsilon^{1/8-\delta}$, so that, with Lemma A.5, the integral of the cubic term in (A.3) from $\Upsilon^{-4/3}$ to $C_\varepsilon^{3/4}\,\varepsilon^{-1/2}$ vanishes as $\varepsilon \to 0$.

This ends the proof of Lemma 7.1.

Before closing, we make the following remark: if $\varepsilon \to 0$, eqn. (7.9) becomes:

$$\frac{d^2p}{d\xi^2} + p + \frac{C_1 p^2}{\xi^{3/8}} + \frac{C_2 p^3}{\xi^{3/4}} = 0 \tag{A.31}$$

to be solved on $[\xi_0, \infty)$. The difficulty in estimating the asymptotic behaviour of the solutions of this equation is that the terms containing powers of p larger than unity do not drop off in an integrable manner as $\xi \to \infty$. For this latter situation, standard theorems are available (Ref.[9],p.91, Ref.[29],p.344), according to which $p(\xi) \to A\cos(\xi + \phi_0)$ as $\xi \to \infty$. According to Lemma 7.1, the nonintegrability does not destroy the ultimate boundedness of the solutions: it only leads to a phase that increases indefinitely as $\xi \to \infty$: $p(\xi) = A\cos(\xi + \phi(\xi) + \phi_0)$, $\phi(\xi) \sim \xi^{1/4}$.

Appendix B: Proof of Lemma 8.2

We call $u_0(\tau)$, $u_\varepsilon(\tau)$ the solutions of (7.6) corresponding to $\varepsilon = 0$ and ε (small); the initial conditions (7.3) are different by $O(\hat{\gamma})$ (cf.(4.33)). On a finite interval of values of τ, $u_0(\tau), u_\varepsilon(\tau)$ and their derivatives are still different by $O(\hat{\gamma})$. Let then τ_0 be such that $\gamma_{oR}(\tau;\hat{\gamma}) > 0$ (cf. eqn.(4.33)), for $\tau > \tau_0$ and $\xi_0 = \xi(\tau_0)$. Let $p_0(\xi), p_\varepsilon(\xi)$ correspond to $u_0(\tau), u_\varepsilon(\tau)$ by (7.8).

The proof consists of the justification of the following statements:

(i) if $\xi \in [\xi_0, a\Upsilon^{-1/3}]$, $|p_\varepsilon(\xi) - p_0(\xi)| \to 0$ as $\varepsilon \to 0$, uniformly with respect to ξ; the same is true for $dp_\varepsilon(\xi)/d\xi$, $dp_0/d\xi$; (ii) $|p_\varepsilon^2(a\Upsilon^{-1/3}) + (dp_\varepsilon/d\xi)^2(a\Upsilon^{-1/3}$

$- R_{oL}^2(\varepsilon)| \to 0$ as $\varepsilon \to 0$. In Appendix A, we showed that $[p_o^2 + (dp_o/d\xi)^2](a\gamma^{-1/3}) \to R_o$ as $\varepsilon \to 0$, so that (i) and (ii) establish the claim concerning $R_{oL}(\varepsilon)$; we show then (iii) $\lim_{\varepsilon \to \varepsilon} \Psi'(\varepsilon) = \Psi'(0)$.

For (i), we subtract from (7.9)(written for p_ε) a similar equation for p_o. With:

$$\Delta p = p_\varepsilon - p_o \tag{B.1}$$

we obtain an equation for Δp, whose linear part is the variational eqn.(8.6), if we let $x(\tau) = p(\tau)\gamma(\ )^{1/2}$ and $\varepsilon = 0$. We use next the two solutions (8.8-9) of (8.6) with $\varepsilon = 0$ (and the factor $\gamma^{-1/2}$ removed) to obtain an integral equation for Δp, using the initial conditions (of $O(\gamma)$) at ξ_o. We use then the estimates (7.10),(7.12) and the fact that, by Lemma 7.1, $p_\varepsilon, dp_\varepsilon/d\xi$ are bounded on $I_\xi(\varepsilon)$, uniformly with respect to ε, to show by a contraction argument that the integral equation admits of a solution with $\|\Delta p\| = O(\gamma^{2/3})$ on $[\xi_o, \gamma^{-1/3}]$. Taking derivatives, one also obtains $\|d/d\xi\,(\Delta p)\| = O(\gamma^{2/3})$ there. Details are given in Ref.[23]; this disposes of (i). Intuitively,because (most of) the solutions of the variational equation with $\xi = 0$ increase like $\xi^{1/4}$, the distance between solutions of(7.9 with initial conditions different by $O(\gamma)$ becomes of order unity only at $\xi \sim \gamma^{-4}$, whereas the effect of the damping is manifest only at $\gamma^{-4/3}$. Thus, we expect (i) to hold at $\xi \sim \gamma^{-1/3}$.

Now, with eqn. (7.24), $\exp(-s(\gamma^{-1/3})) = 1 + O(\gamma^{3/4})$, so that (7.31) implies:

$$R(\gamma^{-1/3};\varepsilon) = R_{oL}(\varepsilon) + O(\gamma^{1/8},\ \gamma \ln 1/\gamma) \tag{B.2}$$

which is statement (ii).

As for (iii), we notice first that (7.32),(7.33) and (8.17) imply that, at $\xi = \gamma^{-1/3}$:

$$\Psi(\gamma^{-1/3};\varepsilon) - \Psi(\varepsilon) = O(\gamma^{1/24}) \tag{B.3}$$

for all $\varepsilon > 0$ and small enough. On the other hand, $\|\Delta p, d\Delta p/d\xi\| = O(\gamma^{2/3})$ implies that:

$$|\phi(\gamma^{-1/3};\varepsilon) - \phi(\gamma^{-1/3};0)| = O(\gamma^{2/3}) \tag{B.4}$$

and it is true that:

$$J(\gamma;\gamma^{-1/3};3/4) - J(0;\gamma^{-1/3};3/4) = O(\gamma^{2/3}) \tag{B.5}$$

Eqns. (B.3-5) imply (iii). This ends the (sketch of the) proof of Lemma 8.2.

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